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Boundary value problems with measures for elliptic equations with singular potentials ☆,☆☆

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Abstract

We study the boundary value problem with Radon measures for nonnegative solutions of $L_V u := -\Delta u + Vu = 0$ in a bounded smooth domain Ω, when V is a locally bounded nonnegative function. Introducing some specific capacity, we give sufficient conditions on a Radon measure μ on $\partial\Omega$ so that the problem can be solved. We study the reduced measure associated to this equation as well as the boundary trace of positive solutions. In Appendix A A. Ancona solves a question raised by M. Marcus and L. Véron concerning the vanishing set of the Poisson kernel of L_V for an important class of potentials V.

Keywords: Laplacian; Poisson potential; Capacities; Singularities; Borel measures; Harnack inequalities

Contents

☆ With Appendix A by Alano Ancona.

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1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and V a locally bounded real valued measurable function defined in Ω. The first question we address is the solvability of the following non-homogeneous Dirichlet problem with a Radon measure for boundary data,

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega, \\ u = \mu & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

Let ϕ be the first (and positive) eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$. By a solution we mean a function $u \in L^1(\Omega)$, such that $Vu \in L^1_\phi$, which satisfies

$$\int_\Omega (-u\Delta\zeta + Vu\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu \tag{1.2}$$

for any function $\zeta \in C_0^1(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty(\Omega)$. When V is a bounded nonnegative function, it is straightforward that there exists a unique solution. However, it is less obvious to find general conditions which allow the solvability for any $\mu \in \mathfrak{M}(\partial\Omega)$, the set of Radon measures on $\partial\Omega$. In order to avoid difficulties due to Fredholm type obstructions, we shall most often assume that V is nonnegative, in which case there exists at most one solution.

Let us denote by K^Ω the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of a measure, that is

$$\mathbb{K}[\mu](x) := \int_{\partial\Omega} K^\Omega(x,y)\,d\mu(y), \quad \forall x \in \Omega. \tag{1.3}$$

We first observe that, when $V \geqslant 0$ and the measure μ satisfies

$$\int_\Omega \mathbb{K}\big[|\mu|\big](x)V(x)\phi(x)\,dx < \infty, \tag{1.4}$$

then problem (1.1) admits a solution. A Radon measure which satisfies (1.4) is called *an admissible measure* and a measure for which a solution exists is called *a good measure*.

We first consider the *subcritical case* which means that the boundary value is solvable for any $\mu \in \mathfrak{M}(\partial\Omega)$. As a first result, we prove that any measure μ is admissible if V is nonnegative and satisfies

$$\sup_{y\in\partial\Omega}\text{ess}\int_{\Omega} K^{\Omega}(x,y)V(x)\phi(x)\,dx < \infty, \tag{1.5}$$

where ϕ is the first positive eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$. Using estimates on the Poisson kernel, this condition is fulfilled if there exists $M>0$ such that for any $y\in\partial\Omega$,

$$\int_0^{D(\Omega)}\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}} \leqslant M \tag{1.6}$$

where $D(\Omega)=\operatorname{diam}(\Omega)$. We give also sufficient conditions which ensure that the boundary value problem (1.1) is stable from the weak*-topology of $\mathfrak{M}(\partial\Omega)$ to $L^1(\Omega)\cap L^1_{V\phi}(\Omega)$. One of the sufficient conditions is that $V\geqslant 0$ satisfies

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}} = 0, \tag{1.7}$$

uniformly with respect to $y\in\partial\Omega$.

In the *supercritical case* problem (1.1) cannot be solved for any $\mu\in\mathfrak{M}(\partial\Omega)$. In order to characterize positive good measures, we introduce a framework of nonlinear analysis which have been used by Dynkin and Kuznetsov (see [11] and references therein) and Marcus and Véron [23,18,24] in their study of the boundary value problems with measures

$$\begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega,\\ u=\mu & \text{in } \partial\Omega, \end{cases} \tag{1.8}$$

where $q>1$. In these works, positive good measures on $\partial\Omega$ are completely characterized by the $C_{2/q,q'}$-Bessel capacity in dimension $N-1$ and the following property:

A measure $\mu\in\mathfrak{M}_+(\partial\Omega)$ is good for problem (1.8) *if and only if it does charge Borel sets with zero $C_{2/q,q'}$-capacity, i.e.*

$$C_{2/q,q'}(E)=0 \quad\Rightarrow\quad \mu(E)=0, \quad \forall E\subset\partial\Omega,\ E \text{ Borel}. \tag{1.9}$$

Moreover, any positive good measure is the limit of an increasing sequence $\{\mu_n\}$ of admissible measures which, in this case, are the positive measures belonging to the Besov space $B_{-2/q,q}(\partial\Omega)$. They also characterize removable sets in terms of $C_{2/q,q'}$-capacity.

In our present work, and always with $V\geqslant 0$, we use a capacity associated to the Poisson kernel K^{Ω} and which belongs to a class studied by Fuglede [13,14]. It is defined by

$$C_V(E) = \sup\big\{\mu(E)\colon\ \mu \in \mathfrak{M}_+(\partial\Omega),\ \mu\big(E^c\big) = 0,\ \big\|V\mathbb{K}[\mu]\big\|_{L^1_\phi} \leqslant 1\big\}, \tag{1.10}$$

for any Borel set $E \subset \partial\Omega$. Furthermore $C_V(E)$ is equal to the value of its dual expression $C_V^*(E)$ defined by

$$C_V^*(E) = \inf\big\{\|f\|_{L^\infty}\colon\ \check{\mathbb{K}}[f] \geqslant 1 \text{ on } E\big\}, \tag{1.11}$$

where

$$\check{\mathbb{K}}[f](y) = \int_\Omega K^\Omega(x,y) f(x) V(x)\phi(x)\,dx, \quad \forall y \in \partial\Omega. \tag{1.12}$$

If E is a compact subset of $\partial\Omega$, this capacity is explicitly given by

$$C_V(E) = C_V^*(E) = \max_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx\right)^{-1}. \tag{1.13}$$

We denote by Z_V the largest set with zero C_V capacity, i.e.

$$Z_V = \left\{y \in \partial\Omega\colon \int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx = \infty\right\}, \tag{1.14}$$

and we prove the following.

1. If $\{\mu_n\}$ is an increasing sequence of positive good measures which converges to a measure μ in the weak*-topology, then μ is a good measure.
2. If $\mu \in \mathfrak{M}_+(\partial\Omega)$ satisfies $\mu(Z_V) = 0$, then μ is a good measure.
3. A good measure μ vanishes on Z_V if and only if there exists an increasing sequence of positive admissible measures which converges to μ in the weak*-topology.

In Section 4 we study relaxation phenomenon in replacing (1.1) by the truncated problem

$$\begin{cases} -\Delta u + V_k u = 0 & \text{in } \Omega,\\ u = \mu & \text{in } \partial\Omega, \end{cases} \tag{1.15}$$

where $\{V_k\}$ is an increasing sequence of positive bounded functions which converges to V locally uniformly in Ω. We adapt to the linear problem some of the principles of the reduced measure. This notion is introduced by Brezis, Marcus and Ponce [7] in the study of the nonlinear Poisson equation

$$-\Delta u + g(u) = \mu \quad \text{in } \Omega \tag{1.16}$$

and extended to the Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega,\\ u = \mu & \text{in } \partial\Omega, \end{cases} \tag{1.17}$$

by Brezis and Ponce [8]. In our construction, problem (1.15) admits a unique solution u_k. The sequence $\{u_k\}$ decreases and converges to some u which satisfies a relaxed boundary value problem

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega, \\ u = \mu^* & \text{in } \partial\Omega. \end{cases} \tag{1.18}$$

The measure μ^* is called the *reduced measure* associated to μ and V. *Note that μ^* is the largest measure for which the problem*

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega, \\ u = \nu \leqslant \mu & \text{in } \partial\Omega, \end{cases} \tag{1.19}$$

admits a solution. This truncation process allows to construct the Poisson kernel K_V^Ω associated to the operator $-\Delta + V$ as being the limit of the decreasing limit of the sequence of kernel functions $\{K_{V_k}^\Omega\}$ associated to $-\Delta + V_k$. The solution $u = u_{\mu^*}$ of (1.18) is expressed by

$$u_{\mu^*}(x) = \int\limits_{\partial\Omega} K_V^\Omega(x,y)\,d\mu(y) = \int\limits_{\partial\Omega} K_V^\Omega(x,y)\,d\mu^*(y), \quad \forall x \in \Omega. \tag{1.20}$$

We define the vanishing set of K_V^Ω by

$$\mathcal{S}ing_V(\Omega) = \big\{y \in \partial\Omega:\ K_V^\Omega(x_0,y) = 0\big\}, \tag{1.21}$$

for some $x_0 \in \Omega$, and thus for any $x \in \Omega$ by Harnack inequality. We prove:

1. $\mathcal{S}ing_V(\Omega) \subset Z_V$.
2. $\mu^* = \mu\chi_{\mathcal{S}ing_V(\Omega)}$.

A challenging open problem is to give conditions on V which imply $\mathcal{S}ing_V(\Omega) = Z_V$.

The last section is devoted to the construction of the boundary trace of positive solutions of

$$-\Delta u + Vu = 0 \quad \text{in } \Omega, \tag{1.22}$$

assuming $V \geqslant 0$. Using results of [20], we defined the regular set $\mathcal{R}(u)$ of the boundary trace of u. This set is a relatively open subset of $\partial\Omega$ and the regular part of the boundary trace is represented by a positive Radon measure μ_u on $\mathcal{R}(u)$. In order to study the singular set of the boundary trace $\mathcal{S}(u) := \partial\Omega \setminus \mathcal{R}(u)$, we adapt the sweeping method introduced by Marcus and Véron in [21] for equation

$$-\Delta u + g(u) = 0 \quad \text{in } \Omega. \tag{1.23}$$

If μ is a good positive measure concentrated on $\mathcal{S}(u)$, and u_μ is the unique solution of (1.1) with boundary data μ, we set $v_\mu = \min\{u, u_\mu\}$. Then v_μ is a positive super solution which admits a positive trace $\gamma_u(\mu) \in \mathfrak{M}_+(\partial\Omega)$. The extended boundary trace $Tr^e(u)$ of u is defined by

$$\nu(u)(E) := Tr^e(u)(E) = \sup\big\{\gamma_u(\mu)(E):\ \mu \text{ good},\ E \subset \partial\Omega,\ E \text{ Borel}\big\}. \tag{1.24}$$

Then $Tr^e(u)$ is a Borel measure on Ω. If we assume moreover that

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\biggl(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\biggr)\frac{dr}{r^{N+1}}=0 \quad \textit{uniformly with respect to } y\in\partial\Omega, \tag{1.25}$$

then $Tr^e(u)$ is a bounded measure and therefore a Radon measure. Finally, if $N=2$ and (1.25) holds, or if $N\geqslant 3$ and there holds

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\biggl(\int_{\Omega\cap B_r(y)} V(x)\bigl(\phi(x)-\epsilon\bigr)_+^2\,dx\biggr)\frac{dr}{r^{N+1}}=0, \tag{1.26}$$

uniformly with respect to $\epsilon\in(0,\epsilon_0]$ and y s.t. $\delta_\Omega(x):=\operatorname{dist}(x,\partial\Omega)=\epsilon$, then $u=u_{\nu(u)}$.

If $V(x)\leqslant v(\phi(x))$ for some v which satisfies

$$\int_0^1 v(t)t\,dt<\infty, \tag{1.27}$$

then Marcus and Véron proved in [20] that $u=u_{\nu_u}$. Actually, when V has such a geometric form, the assumptions (1.25)–(1.26) and (1.27) are equivalent.

Appendix A, written by A. Ancona, answers a question raised by M. Marcus and L. Véron in 2005 about the vanishing set of K_V when V is nonnegative and $\delta_\Omega^2 V$ is uniformly bounded. Such potentials play a very important role in the description of the fine trace of semilinear elliptic equations as in (1.8): actually, for such equations, $V=u^{q-1}$ satisfies this upper estimate as a consequence of Keller–Osserman estimate. The following result is proved:

Let $y\in\partial\Omega$ and $C_{\epsilon,y}:=\{x\in\Omega\colon\ \delta_\Omega(x)\geqslant\epsilon|x-y|\}$ for $0<\epsilon<1$. If

$$\int_{C_{\epsilon,y}}\frac{V(x)\,dx}{|x-y|^{N-2}}=\infty, \tag{1.28}$$

for some $\epsilon>0$, then $y\in\mathcal{S}ing_V(\Omega)$.

2. The subcritical case

In the sequel Ω is a bounded smooth domain in $\mathbb{R}^N$ and $V\in L^\infty_{loc}$. We denote by ϕ the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$, $\phi>0$ with the corresponding eigenvalue λ, by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}_+(\partial\Omega)$ its positive cone. For any positive Radon measure on $\partial\Omega$, we shall denote by the same symbol the corresponding outer regular bounded Borel measure. Conversely, for any outer regular bounded Borel μ, we denote by the same expression μ the Radon measure defined on $C(\partial\Omega)$ by

$$\zeta\mapsto\mu(\zeta)=\int_{\partial\Omega}\zeta\,d\mu.$$

If $\mu \in \mathfrak{M}(\partial\Omega)$, we are concerned with the following problem

$$\begin{cases} -\Delta u + Vu = 0 & \text{in } \Omega, \\ u = \mu & \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

Definition 2.1. Let $\mu \in \mathfrak{M}(\partial\Omega)$. We say that u is a weak solution of (2.1), if $u \in L^1(\Omega)$, $Vu \in L^1_\phi(\Omega)$ and, for any $\zeta \in C^1_0(\overline{\Omega})$ with $\Delta\zeta \in L^\infty(\Omega)$, there holds

$$\int_\Omega (-u\Delta\zeta + Vu\zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu. \tag{2.2}$$

In the sequel we refer to [15] for the classical regularity theory of solutions of second order elliptic equations and we put

$$T(\Omega) := \{\zeta \in C^1_0(\overline{\Omega}) \text{ such that } \Delta\zeta \in L^\infty(\Omega)\}.$$

We recall the following estimates obtained by Brezis [6] (see [28] for a detailed proof).

Proposition 2.2. *Let $\mu \in L^1(\partial\Omega)$ and u be a weak solution of problem* (2.1). *Then there holds*

$$\|u\|_{L^1(\Omega)} + \|V_+u\|_{L^1_\phi(\Omega)} \leqslant \|V_-u\|_{L^1_\phi(\Omega)} + c\|\mu\|_{L^1(\partial\Omega)}, \tag{2.3}$$

$$\int_\Omega \big(-|u|\Delta\zeta + V|u|\zeta\big)\,dx \leqslant -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}|\mu|\,dS \tag{2.4}$$

and

$$\int_\Omega (-u_+\Delta\zeta + Vu_+\zeta)\,dx \leqslant -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\mu_+\,dS, \tag{2.5}$$

for all $\zeta \in T(\Omega)$, $\zeta \geqslant 0$.

We denote by $K^\Omega(x,y)$ the Poisson kernel in Ω and by $\mathbb{K}[\mu]$ the Poisson potential of $\mu \in \mathfrak{M}(\partial\Omega)$ defined by

$$\mathbb{K}[\mu](x) = \int_{\partial\Omega} K^\Omega(x,y)\,d\mu(y), \quad \forall x \in \Omega. \tag{2.6}$$

Definition 2.3. A measure μ on $\partial\Omega$ is **admissible** if

$$\int_\Omega \mathbb{K}\big[|\mu|\big](x)\big|V(x)\big|\phi(x)\,dx < \infty. \tag{2.7}$$

It is **good** if problem (2.1) admits a weak solution.

We notice that, if there exists at least one admissible positive measure μ, then

$$\int_{\Omega} V(x)\phi^2(x)\,dx < \infty. \tag{2.8}$$

Theorem 2.4. *Assume $V \geqslant 0$, then problem* (2.1) *admits at most one solution. Furthermore, if μ is admissible, then there exists a unique solution that we denote u_μ.*

Proof. Uniqueness follows from (2.3). For existence we can assume $\mu \geqslant 0$. For any $k \in \mathbb{N}_*$ set $V_k = \inf\{V, k\}$ and denote by $u := u_k$ the solution of

$$\begin{cases} -\Delta u + V_k(x)u = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Then $0 \leqslant u_k \leqslant \mathbb{K}[\mu]$. By the maximum principle, u_k is decreasing and converges to some u, and

$$0 \leqslant V_k u_k \leqslant V\mathbb{K}[\mu].$$

Thus, by dominated convergence theorem $V_k u_k \to Vu$ in L^1_ϕ. Setting $\zeta \in T(\Omega)$ and letting k tend to infinity in equality

$$\int_{\Omega} (-u_k\Delta\zeta + V_k u_k \zeta)\,dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu, \tag{2.10}$$

implies that u satisfies (2.2). □

Remark. If V changes sign, we can put $\tilde{u} = u + \mathbb{K}[\mu]$. Then (2.1) is equivalent to

$$\begin{cases} -\Delta\tilde{u} + V\tilde{u} = V\mathbb{K}[\mu] & \text{in } \Omega, \\ \tilde{u} = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.11}$$

This is a Fredholm type problem (at least if the operator $\phi \mapsto R(v) := (-\Delta)^{-1}(V\phi)$ is compact in $L^1_\phi(\Omega)$). Existence will be ensured by orthogonality conditions.

If we assume that $V \geqslant 0$ and

$$\int_{\Omega} K^{\Omega}(x,y)V(x)\phi(x)\,dx < \infty, \tag{2.12}$$

for some $y \in \partial\Omega$, then δ_y is admissible. The following result yields to the solvability of (2.1) for any $\mu \in \mathfrak{M}_+(\Omega)$.

Proposition 2.5. *Assume $V \geqslant 0$ and the integrals* (2.12) *are bounded uniformly with respect to $y \in \partial\Omega$. Then any measure on $\partial\Omega$ is admissible.*

Proof. If M is the upper bound of these integrals and $\mu \in \mathfrak{M}_+(\partial\Omega)$, we have

$$\int_\Omega \mathbb{K}[\mu](x)V(x)\phi(x)\,dx = \int_{\partial\Omega}\left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx\right)d\mu(y) \leqslant M\mu(\partial\Omega), \tag{2.13}$$

by Fubini's theorem. Thus μ is admissible. □

Remark. Since the Poisson kernel in Ω satisfies the two-sided estimate

$$c^{-1}\frac{\phi(x)}{|x-y|^N} \leqslant K^\Omega(x,y) \leqslant c\frac{\phi(x)}{|x-y|^N}, \quad \forall(x,y) \in \Omega\times\partial\Omega, \tag{2.14}$$

for some $c > 0$, assumption (2.12) is equivalent to

$$\int_\Omega \frac{V(x)\phi^2(x)}{|x-y|^N}\,dx < \infty. \tag{2.15}$$

This implies (2.8) in particular. If we set $D_y = \max\{|x-y|:\ x \in \Omega\}$, then

$$\begin{aligned}
&\int_\Omega \frac{V(x)\phi^2(x)}{|x-y|^N}\,dx \\
&\quad= \int_0^{D_y}\left(\int_{\{x\in\Omega:\ |x-y|=r\}} V(x)\phi^2(x)\,dS_r(x)\right)\frac{dr}{r^N} \\
&\quad= \lim_{\epsilon\to 0}\left(\left[r^{-N}\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right]_\epsilon^{D_y} + N\int_\epsilon^{D_y}\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}}\right)
\end{aligned}$$

(both quantity may be infinite). Thus, if we assume

$$\int_0^{D_y}\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}} < \infty, \tag{2.16}$$

there holds

$$\liminf_{\epsilon\to 0}\epsilon^{-N}\int_{\Omega\cap B_\epsilon(y)} V(x)\phi^2(x)\,dS = 0. \tag{2.17}$$

Consequently

$$\int_\Omega \frac{V(x)\phi^2(x)}{|x-y|^N}\,dx = D_y^{-N}\int_\Omega V(x)\phi^2(x)\,dx + N\int_0^{D_y}\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}}. \tag{2.18}$$

Therefore (2.12) holds and δ_y is admissible.

As a natural extension of Proposition 2.5, we have the following stability result.

Theorem 2.6. *Assume $V \geqslant 0$ and*

$$\lim_{\substack{E \text{ Borel} \\ |E|\to 0}} \int_E K^\Omega(x,y)V(x)\phi(x)\,dx = 0 \quad \textit{uniformly with respect to } y\in\partial\Omega. \tag{2.19}$$

If μ_n is a sequence of positive Radon measures on $\partial\Omega$ converging to μ in the weak-topology, then u_{μ_n} converges to u_μ in $L^1(\Omega)\cap L^1_{V\phi}(\Omega)$ and locally uniformly in Ω.*

Proof. We put $u_{\mu_n} := u_n$. By the maximum principle $0 \leqslant u_n \leqslant \mathbb{K}[\mu_n]$. Furthermore, it follows from (2.3) that

$$\|u_n\|_{L^1(\Omega)} + \|Vu_n\|_{L^1_\phi(\Omega)} \leqslant c\|\mu_n\|_{L^1(\partial\Omega)} \leqslant C. \tag{2.20}$$

Since $-\Delta u_n$ is bounded in $L^1_\phi(\Omega)$, the sequence $\{u_n\}$ is relatively compact in $L^1(\Omega)$ by the regularity theory for elliptic equations. Therefore, there exist a subsequence u_{n_k} and some function $u\in L^1(\Omega)$ with $Vu\in L^1_\phi(\Omega)$ such that u_{n_k} converges to u in $L^1(\Omega)$, almost everywhere on Ω and locally uniformly in Ω since $V\in L^\infty_{loc}(\Omega)$. The main question is to prove the convergence of Vu_{n_k} in $L^1_\phi(\Omega)$. If $E\subset\Omega$ is any Borel set, there holds

$$\begin{aligned}\int_E u_nV(x)\phi(x)\,dx &\leqslant \int_E \mathbb{K}[\mu_n]V(x)\phi(x)\,dx \\ &\leqslant \int_{\partial\Omega}\left(\int_E K^\Omega(x,y)V(x)\phi(x)\,dx\right)d\mu_n(y) \\ &\leqslant M_n \max_{y\in\partial\Omega}\int_E K^\Omega(x,y)V(x)\phi(x)\,dx,\end{aligned}$$

where $M_n := \mu_n(\partial\Omega)$. Thus

$$\int_E u_nV(x)\phi(x)\,dx \leqslant M_n \max_{y\in\partial\Omega}\int_E K^\Omega(x,y)V(x)\phi(x)\,dx. \tag{2.21}$$

Then, by (2.19),

$$\lim_{|E|\to 0}\int_E u_n V(x)\phi(x)\,dx = 0.$$

As a consequence the set of function $\{u_n\phi V\}$ is uniformly integrable. By Vitali's theorem $Vu_{n_k} \to Vu$ in $L^1_\phi(\Omega)$. Since

$$\int_\Omega (-u_n\Delta\zeta + Vu_n\zeta)\,dx = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu_n, \tag{2.22}$$

for any $\zeta \in T(\Omega)$, the function u satisfies (2.2). □

Assumption (2.19) may be difficult to verify and the following result gives an easier formulation.

Proposition 2.7. *Assume $V \geqslant 0$ satisfies*

$$\lim_{\epsilon\to 0}\int_0^\epsilon\left(\int_{\Omega\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}} = 0 \quad \textit{uniformly with respect to } y\in\partial\Omega. \tag{2.23}$$

Then (2.19) *holds.*

Proof. If $E \subset \Omega$ is a Borel set and $\delta > 0$, we put $E_\delta = E \cap B_\delta(y)$ and $E^c_\delta = E \setminus E_\delta$. Then

$$\int_E \frac{V(x)\phi^2(x)}{|x-y|^N}\,dx = \int_{E_\delta}\frac{V(x)\phi^2(x)}{|x-y|^N}\,dx + \int_{E^c_\delta}\frac{V(x)\phi^2(x)}{|x-y|^N}\,dx.$$

Clearly

$$\int_{E^c_\delta}\frac{V(x)\phi^2(x)}{|x-y|^N}\,dx \leqslant \delta^{-N}\int_{E.} V(x)\phi^2(x)\,dx. \tag{2.24}$$

Since (2.16) holds for any $y \in \partial\Omega$, (2.18) implies

$$\int_{E_\delta}\frac{V(x)\phi^2(x)}{|x-y|^N}\,dx = \delta^{-N}\int_{E_\delta} V(x)\phi^2(x)\,dx + N\int_0^\delta\left(\int_{E\cap B_r(y)} V(x)\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}}. \tag{2.25}$$

Using (2.23), for any $\epsilon > 0$, there exists $s_0 > 0$ such that for any $s > 0$ and $y \in \partial\Omega$

$$s \leqslant s_0 \quad \Rightarrow \quad N\int_0^s \left(\int_{B_r(y)} V(x)\phi^2(x)\,dx \right) \frac{dr}{r^{N+1}} \leqslant \epsilon/2.$$

We fix $\delta = s_0$. Since (2.8) holds,

$$\lim_{\substack{E \text{ Borel} \\ |E| \to 0}} \int_E V(x)\phi^2(x)\,dx = 0. \tag{2.26}$$

Then there exists $\eta > 0$ such that for any Borel set $E \subset \Omega$,

$$|E| \leqslant \eta \quad \Rightarrow \quad \int_E V(x)\phi^2(x)\,dx \leqslant s_0^N \epsilon/4.$$

Thus

$$\int_E \frac{V(x)\phi^2(x)}{|x-y|^N}\,dx \leqslant \epsilon.$$

This implies the claim by (2.14). $\square$

An assumption which is used in [20, Lemma 7.4] in order to prove the existence of a boundary trace of any positive solution of (1.22) is that there exists some nonnegative measurable function v defined on $\mathbb{R}_+$ such that

$$\left|V(x)\right| \leqslant v\big(\phi(x)\big), \quad \forall x \in \Omega \quad \text{and} \quad \int_0^s t v(t)\,dt < \infty, \quad \forall s > 0. \tag{2.27}$$

In the next result we show that condition (2.27) implies (2.19).

Proposition 2.8. *Assume V satisfies* (2.27). *Then*

$$\lim_{\substack{E \text{ Borel} \\ |E| \to 0}} \int_E K^\Omega(x,y)\left|V(x)\right|\phi(x)\,dx = 0 \quad \textit{uniformly with respect to } y \in \partial\Omega. \tag{2.28}$$

Proof. Since $\partial\Omega$ is C^2, there exists $\epsilon_0 > 0$ such that for any $x \in \Omega$ satisfying $\phi(x) \leqslant \epsilon_0$, there exists a unique $\sigma(x) \in \partial\Omega$ such that $|x - \sigma(x)| = \phi(x)$. We use (2.23) in Proposition 2.7 under the equivalent form

$$\lim_{\epsilon \to 0} \int_0^\epsilon \left(\int_{\Omega \cap C_r(y)} \left|V(x)\right|\phi^2(x)\,dx \right) \frac{dr}{r^{N+1}} = 0 \quad \text{uniformly with respect to } y \in \partial\Omega, \tag{2.29}$$

in which we have replaced $B_r(y)$ by the cylinder $C_r(y) := \{x \in \Omega\colon\ \phi(x) < r, |\sigma(x) - y| < r\}$. Then

$$\int_0^{\epsilon}\left(\int_{\Omega\cap C_r(y)} |V(x)|\phi^2(x)\,dx\right)\frac{dr}{r^{N+1}} \leqslant c\int_0^{\epsilon}\left(\int_0^r v(t)t^2\,dt\right)\frac{dr}{r^2}$$
$$\leqslant c\int_0^{\epsilon} v(t)\left(1-\frac{t}{\epsilon}\right)t\,dt$$
$$\leqslant c\int_0^{\epsilon} v(t)t\,dt.$$

Thus (2.23) holds. □

3. The capacitary approach

Throughout this section V is a locally bounded nonnegative and measurable function defined on Ω. We assume that there exists a positive measure μ_0 on $\partial\Omega$ such that

$$\int_{\Omega} \mathbb{K}[\mu_0]V(x)\phi(x)\,dx = \mathcal{E}(1,\mu_0) < \infty. \tag{3.1}$$

Definition 3.1. If $\mu \in \mathfrak{M}_+(\partial\Omega)$ and f is a nonnegative measurable function defined in Ω such that

$$(x,y) \mapsto \mathbb{K}[\mu](y)f(x)V(x)\phi(x) \in L^1(\Omega\times\partial\Omega; dx\otimes d\mu),$$

we set

$$\mathcal{E}(f,\mu) = \int_{\Omega}\left(\int_{\partial\Omega} K^{\Omega}(x,y)\,d\mu(y)\right)f(x)V(x)\phi(x)\,dx. \tag{3.2}$$

If we put

$$\check{\mathbb{K}}_V[f](y) = \int_{\Omega} K^{\Omega}(x,y)f(x)V(x)\phi(x)\,dx, \tag{3.3}$$

then, by Fubini's theorem, $\check{\mathbb{K}}_V[f] < \infty$, μ-almost everywhere on $\partial\Omega$ and

$$\mathcal{E}(f,\mu) = \int_{\partial\Omega}\left(\int_{\Omega} K^{\Omega}(x,y)f(x)V(x)\phi(x)\,dx\right)d\mu(y). \tag{3.4}$$

Proposition 3.2. *Let f be fixed. Then*:

(a) $y \mapsto \check{\mathbb{K}}_V[f](y)$ *is lower semicontinuous on* $\partial\Omega$.
(b) $\mu \mapsto \mathcal{E}(f,\mu)$ *is lower semicontinuous on* $\mathfrak{M}_+(\partial\Omega)$ *in the weak***-topology.*

Proof. Since $y \mapsto K^\Omega(x,y)$ is continuous, statement (a) follows by Fatou's lemma. If μ_n is a sequence in $\mathfrak{M}_+(\partial\Omega)$ converging to some μ in the weak*-topology, then $\mathbb{K}[\mu_n]$ converges to $\mathbb{K}[\mu]$ everywhere in Ω. By Fatou's lemma

$$\mathcal{E}(f,\mu) \leqslant \liminf_{n\to\infty} \int_\Omega \mathbb{K}[\mu_n](x) f(x) V(x)\phi(x)\,dx = \liminf_{n\to\infty} \mathcal{E}(f,\mu_n). \qquad \square$$

Notice that if $V\phi f \in L^p(\Omega)$, for $p > N$, then $\mathbb{G}[Vf\phi] \in C^1(\overline{\Omega})$ and

$$\check{\mathbb{K}}[f](y) := \int_\Omega K^\Omega(x,y)V(x)f(x)\phi(x)\,dx = -\frac{\partial}{\partial \mathbf{n}}\mathbb{G}[Vf\phi](y). \tag{3.5}$$

This is in particular the case if f has compact support in Ω.

Definition 3.3. We denote by $\mathfrak{M}^V(\partial\Omega)$ the set of all measures μ on $\partial\Omega$ such that $V\mathbb{K}[\mu] \in L^1_\phi(\Omega)$. If μ is such a measure, we denote

$$\|\mu\|_{\mathfrak{M}^V} = \int_\Omega \big|\mathbb{K}[\mu](x)\big| V(x)\phi(x)\,dx = \big\|V\mathbb{K}[\mu]\big\|_{L^1_\phi}. \tag{3.6}$$

Clearly $\|\,.\,\|_{\mathfrak{M}^V}$ is a norm. The space $\mathfrak{M}^V(\partial\Omega)$ is not complete but its positive cone $\mathfrak{M}^V_+(\partial\Omega)$ is complete. If $E \subset \partial\Omega$ is a Borel subset, we put

$$\mathfrak{M}_+(E) = \big\{\mu \in \mathfrak{M}_+(\partial\Omega):\ \mu\big(E^c\big) = 0\big\} \quad \text{and} \quad \mathfrak{M}^V_+(E) = \mathfrak{M}_+(E) \cap \mathfrak{M}^V(\partial\Omega).$$

Definition 3.4. If $E \subset \partial\Omega$ is any Borel subset we set

$$C_V(E) := \sup\big\{\mu(E):\ \mu \in \mathfrak{M}^V_+(E),\ \|\mu\|_{\mathfrak{M}^V} \leqslant 1\big\}. \tag{3.7}$$

We notice that (3.7) is equivalent to

$$C_V(E) := \sup\left\{\frac{\mu(E)}{\|\mu\|_{\mathfrak{M}^V}}:\ \mu \in \mathfrak{M}^V_+(E)\right\}. \tag{3.8}$$

Proposition 3.5. *The set function C_V satisfies*

$$C_V(E) \leqslant \sup_{y\in E}\left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx\right)^{-1}, \quad \forall E \subset \partial\Omega,\ E \text{ Borel}, \tag{3.9}$$

and equality holds in (3.9) *if E is compact. Moreover,*

$$C_V(E_1 \cup E_2) = \sup\{C_V(E_1), C_V(E_2)\}, \quad \forall E_i \subset \partial\Omega,\ E_i \text{ Borel}. \tag{3.10}$$

Proof. Notice that $E \mapsto C_V(E)$ is a nondecreasing set function for the inclusion relation and that (3.7) implies

$$\mu(E) \leqslant C_V(E)\|\mu\|_{\mathfrak{M}^V}, \quad \forall \mu \in \mathfrak{M}_+^V(E). \tag{3.11}$$

Let $E \subset \partial\Omega$ be a Borel set and $\mu \in \mathfrak{M}_+(E)$. Then

$$\begin{aligned}\|\mu\|_{\mathfrak{M}^V} &= \int_E \left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx \right) d\mu(y) \\ &\geqslant \mu(E) \inf_{y\in E} \int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx.\end{aligned}$$

Using (3.7) we derive

$$C_V(E) \leqslant \sup_{y\in E} \left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx \right)^{-1}. \tag{3.12}$$

If E is compact, there exists $y_0 \in E$ such that

$$\inf_{y\in E} \int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx = \int_\Omega K^\Omega(x,y_0)V(x)\phi(x)\,dx,$$

since $y \mapsto \check{\mathbb{K}}[1](y)$ is l.s.c.. Thus

$$\|\delta_{y_0}\|_{\mathfrak{M}^V} = \delta_{y_0}(E) \int_\Omega K^\Omega(x,y_0)V(x)\phi(x)\,dx$$

and

$$C_V(E) \geqslant \frac{\delta_{y_0}(E)}{\|\delta_{y_0}\|_{\mathfrak{M}^V}} = \sup_{y\in E} \left(\int_\Omega K^\Omega(x,y)V(x)\phi(x)\,dx \right)^{-1}.$$

Therefore equality holds in (3.9). Identity (3.10) follows (3.9) when there is equality. Moreover it holds if E_1 and E_2 are two arbitrary compact sets. Since C_V is eventually an inner regular capacity (i.e. $C_V(E) = \sup\{C_V(K)\colon\ K \subset E,\ K \text{ compact}\}$) it holds for any Borel set. However we give below a self-contained proof. If E_1 and E_2 be two disjoint Borel subsets of $\partial\Omega$, for any $\epsilon > 0$ there exists $\mu \in \mathfrak{M}_+^V(E_1 \cup E_2)$ such that

$$\frac{\mu(E_1) + \mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}} \leqslant C_V(E_1 \cup E_2) \leqslant \frac{\mu(E_1) + \mu(E_2)}{\|\mu\|_{\mathfrak{M}^V}} + \epsilon.$$

Set $\mu_i = \chi_{E_i}\mu$. Then $\mu_i \in \mathfrak{M}_+^V(E_i)$ and $\|\mu\|_{\mathfrak{M}^V} = \|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}$. By (3.11)

$$\begin{aligned} C_V(E_1 \cup E_2) \leqslant{} & \frac{\|\mu_1\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}} C_V(E_1) \\ & + \frac{\|\mu_2\|_{\mathfrak{M}^V}}{\|\mu_1\|_{\mathfrak{M}^V} + \|\mu_2\|_{\mathfrak{M}^V}} C_V(E_2) + \epsilon. \end{aligned} \tag{3.13}$$

This implies that there exists $\theta \in [0, 1]$ such that

$$C_V(E_1 \cup E_2) \leqslant \theta C_V(E_1) + (1-\theta) C_V(E_2) \leqslant \max\{C_V(E_1), C_V(E_2)\}. \tag{3.14}$$

Since $C_V(E_1 \cup E_2) \geqslant \max\{C_V(E_1), C_V(E_2)\}$ as C_V is increasing,

$$E_1 \cap E_2 = \emptyset \quad \Rightarrow \quad C_V(E_1 \cup E_2) = \max\{C_V(E_1), C_V(E_2)\}. \tag{3.15}$$

If $E_1 \cap E_2 \neq \emptyset$, then $E_1 \cup E_2 = E_1 \cup (E_2 \cap E_1^c)$ and therefore

$$C_V(E_1 \cup E_2) = \max\{C_V(E_1), C_V(E_2 \cap E_1^c)\} \leqslant \max\{C_V(E_1), C_V(E_2)\}.$$

Using again (3.8) we derive (3.10). □

The following set function is the dual expression of $C_V(E)$.

Definition 3.6. For any Borel set $E \subset \partial\Omega$, we set

$$C_V^*(E) := \inf\{\|f\|_{L^\infty}\colon \check{\mathbb{K}}[f](y) \geqslant 1,\ \forall y \in E\}. \tag{3.16}$$

The next result is stated in [14, p. 922] using minimax theorem and the fact that K^Ω is lower semicontinuous in $\Omega \times \partial\Omega$. Although the proof is not explicited, a simple adaptation of the proof of [1, Theorem 2.5.1] leads to the result.

Proposition 3.7. *For any compact set $E \subset \partial\Omega$,*

$$C_V(E) = C_V^*(E). \tag{3.17}$$

In the same paper [14], formula (3.9) with equality is claimed (if E is compact).

Theorem 3.8. *If $\{\mu_n\}$ is an increasing sequence of good measures converging to some measure μ in the weak*-topology, then μ is good.*

Proof. We use formulation (4.10). We take for test function the function η solution of

$$\begin{cases} -\Delta\eta = 1 & \text{in } \Omega, \\ \eta = 0 & \text{on } \Omega, \end{cases} \tag{3.18}$$

there holds

$$\int_{\Omega}(1+V)u_{\mu_n}\eta\,dx = -\int_{\partial\Omega}\frac{\partial\eta}{\partial\mathbf{n}}\,d\mu_n \leqslant c^{-1}\mu_n(\partial\Omega) \leqslant c^{-1}\mu(\partial\Omega)$$

where $c > 0$ is such that

$$c^{-1} \geqslant -\frac{\partial\eta}{\partial\mathbf{n}} \geqslant c \quad \text{on } \partial\Omega.$$

Since $\{u_{\mu_n}\}$ is increasing and $\eta \leqslant c\phi$ by Hopf boundary lemma, we can let $n \to \infty$ by the monotone convergence theorem. If $u := \lim_{n\to\infty} u_{\mu_n}$, we obtain

$$\int_{\Omega}(1+V)u\eta\,dx \leqslant c^{-1}\mu(\partial\Omega).$$

Thus u and $\phi V u$ are in $L^1(\Omega)$. Next, if $\zeta \in C_0^1(\overline{\Omega}) \cap C^{1,1}(\overline{\Omega})$, then $u_{\mu_n}|\Delta\zeta| \leqslant C u_{\mu_n}$ and $V u_{\mu_n}|\zeta| \leqslant C V u_{\mu_n}\eta$. Because the sequence $\{u_{\mu_n}\}$ and $\{V u_{\mu_n}\eta\}$ are uniformly integrable, the same holds for $\{u_{\mu_n}\Delta\zeta\}$ and $\{V u_{\mu_n}\zeta\}$. Considering

$$\int_{\Omega}(-u_{\mu_n}\Delta\zeta + V u_{\mu_n}\zeta)\,dx = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu_n,$$

it follows by Vitali's theorem,

$$\int_{\Omega}(-u\Delta\zeta + V u\zeta)\,dx = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}\,d\mu.$$

Thus μ is a good measure. $\square$

We define the *singular boundary set* Z_V by

$$Z_V = \left\{ y \in \partial\Omega\colon \int_{\Omega} K^{\Omega}(x,y)V(x)\phi(x)\,dx = \infty \right\}. \tag{3.19}$$

Since $\check{\mathbb{K}}[1]$ is l.s.c., it is a Borel function and Z_V is a Borel set. The next result characterizes the good measures.

Proposition 3.9. *Let μ be an admissible positive measure. Then $\mu(Z_V) = 0$.*

Proof. If $K \subset Z_V$ is compact, $\mu_K = \chi_K\mu$ is admissible, thus, by Fubini theorem

$$\|\mu_K\|_{\mathfrak{M}^V} = \int_K \left(\int_{\Omega} K^{\Omega}(x,y)V(x)\phi(x)\,dx \right) d\mu(y) < \infty.$$

Since

$$\int_{\Omega} K^{\Omega}(x,y)V(x)\phi(x)\,dx \equiv \infty, \quad \forall y \in K$$

it follows that $\mu(K)=0$. This implies $\mu(Z_V)=0$ by regularity. $\square$

Theorem 3.10. *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that*

$$\mu(Z_V)=0. \tag{3.20}$$

Then μ is good.

Proof. Since $\check{\mathbb{K}}[1]$ is l.s.c., for any $n \in \mathbb{N}_*$,

$$K_n := \left\{y \in \partial\Omega\colon\ \check{\mathbb{K}}[1](y) \leqslant n\right\}$$

is a compact subset of $\partial\Omega$. Furthermore $K_n \cap Z_V = \emptyset$ and $\bigcup K_n = Z_V^c$. Let $\mu_n = \chi_{K_n}\mu$, then

$$\mathcal{E}(1,\mu_n) = \int_{\Omega} \mathbb{K}[\mu_n]V(x)\phi(x)\,dx \leqslant n\mu_n(K_n). \tag{3.21}$$

Therefore μ_n is admissible. By the monotone convergence theorem, $\mu_n \uparrow \chi_{Z_V{}^c}\mu$ and by Theorem 3.8, $\chi_{Z_V{}^c}\mu$ is good. Since (3.20) holds, $\chi_{Z_V{}^c}\mu = \mu$, which ends the proof. $\square$

The full characterization of the good measures in the general case appears to be difficult without any further assumptions on V. However the following holds

Theorem 3.11. *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be a good measure. The following assertions are equivalent*:

(i) $\mu(Z_V)=0$.
(ii) *There exists an increasing sequence of admissible measures $\{\mu_n\}$ which converges to μ in the weak*-topology.*

Proof. If (i) holds, it follows from the proof of Theorem 3.10 that the sequence $\{\mu_n\}$ increases and converges to μ. If (ii) holds, any admissible measure μ_n vanishes on Z_V by Proposition 3.9. Since $\mu_n \leqslant \mu$, there exists an increasing sequence of μ-integrable functions h_n such that $\mu_n = h_n\mu$. Then $\mu_n(Z_V)$ increases to $\mu(Z_V)$ by the monotone convergence theorem. The conclusion follows from the fact that $\mu_n(Z_V)=0$. $\square$

4. Representation formula and reduced measures

We recall the construction of the Poisson kernel for $-\Delta + V$: if we look for a solution of

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega, \\ v = \nu & \text{in } \partial\Omega, \end{cases} \tag{4.1}$$

where $\nu \in \mathfrak{M}(\partial\Omega)$, $V \geqslant 0$, $V \in L^{\infty}_{loc}(\Omega)$, we can consider an increasing sequence of smooth domains Ω_n such that $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\bigcup_n \Omega_n = \bigcup_n \overline{\Omega}_n = \Omega$. For each of these domains, denote by $K^{\Omega}_{V\chi_{\Omega_n}}$ the Poisson kernel of $-\Delta+V\chi_{\Omega_n}$ in Ω and by $\mathbb{K}_{V\chi_{\Omega_n}}[.]$ the corresponding operator. We denote by $K^{\Omega}:=K^{\Omega}_0$ the Poisson kernel in Ω and by $\mathbb{K}[.]$ the Poisson operator in Ω. Then the solution $v:=v_n$ of

$$\begin{cases}-\Delta v+V\chi_{\Omega_n}v=0 & \text{in } \Omega,\\ v=\nu & \text{in } \partial\Omega,\end{cases} \tag{4.2}$$

is expressed by

$$v_n(x)=\int\limits_{\partial\Omega}K^{\Omega}_{V\chi_{\Omega_n}}(x,y)\,d\nu(y)=\mathbb{K}_{V\chi_{\Omega_n}}[\nu](x). \tag{4.3}$$

If G^{Ω} is the Green kernel of $-\Delta$ in Ω and $\mathbb{G}[.]$ the corresponding Green operator, (4.3) is equivalent to

$$v_n(x)+\int\limits_{\Omega}G^{\Omega}(x,y)(V\chi_{\Omega_n}v_n)(y)\,dy=\int\limits_{\partial\Omega}K^{\Omega}(x,y)\,d\nu(y), \tag{4.4}$$

equivalently

$$v_n+\mathbb{G}[V\chi_{\Omega_n}v_n]=\mathbb{K}[\nu].$$

Notice that this equality is equivalent to the weak formulation of problem (4.2): for any $\zeta\in T(\Omega)$, there holds

$$\int\limits_{\Omega}(-v_n\Delta\zeta+V\chi_{\Omega_n}v_n\zeta)\,dx=-\int\limits_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}}\,d\nu. \tag{4.5}$$

Since $n\mapsto K^{\Omega}_{V\chi_{\Omega_n}}$ is decreasing, the sequence $\{v_n\}$ inherits this property and there exists

$$\lim_{n\to\infty}K^{\Omega}_{V\chi_{\Omega_n}}(x,y)=K^{\Omega}_{V}(x,y). \tag{4.6}$$

By the monotone convergence theorem,

$$\lim_{n\to\infty}v_n(x)=v(x)=\int\limits_{\partial\Omega}K^{\Omega}_{V}(x,y)\,d\nu(y). \tag{4.7}$$

By Fatou's theorem

$$\int\limits_{\Omega}G^{\Omega}(x,y)V(y)v(y)\,dy\leqslant\liminf_{n\to\infty}\int\limits_{\Omega}G^{\Omega}(x,y)(V\chi_{\Omega_n}v_n)(y)\,dy, \tag{4.8}$$

and thus,

$$v(x) + \int_{\Omega} G^{\Omega}(x, y)V(y)v(y)\,dy \leqslant \mathbb{K}[\nu](x), \quad \forall x \in \Omega. \tag{4.9}$$

Now the main question is to know whether v keeps the boundary value ν. Equivalently, whether the equality holds in (4.8) with lim instead of lim inf, and therefore in (4.9). This question is associated to the notion of reduced measured in the sense of Brezis–Marcus–Ponce: since $Vv \in L^1_\phi(\Omega)$ and

$$-\Delta v + V(x)v = 0 \quad \text{in } \Omega \tag{4.10}$$

holds, the function $v + \mathbb{G}[Vv]$ is positive and harmonic in Ω. Thus it admits a boundary trace $\nu^* \in \mathfrak{M}_+(\partial\Omega)$ and

$$v + \mathbb{G}[Vv] = \mathbb{K}\big[\nu^*\big]. \tag{4.11}$$

Equivalently v satisfies the relaxed problem

$$\begin{cases} -\Delta v + V(x)v = 0 & \text{in } \Omega, \\ v = \nu^* & \text{in } \partial\Omega, \end{cases} \tag{4.12}$$

and thus $v = u_{\nu^*}$. Noticed that $\nu^* \leqslant \nu$ and the mapping $\nu \mapsto \nu^*$ is nondecreasing.

Definition 4.1. The measure ν^* is the *reduced measure* associated to ν.

Proposition 4.2. *There holds $\mathbb{K}_V[\nu] = \mathbb{K}_V[\nu^*]$. Furthermore the reduced measure ν^* is the largest measure for which the following problem*

$$\begin{cases} -\Delta v + V(x)v = 0 & \textit{in } \Omega, \\ \lambda \in \mathfrak{M}_+(\partial\Omega), \quad \lambda \leqslant \nu, \quad v = \lambda & \textit{in } \partial\Omega, \end{cases} \tag{4.13}$$

admits a solution.

Proof. The first assertion follows from the fact that $v = \mathbb{K}_V[\nu]$ by (4.6) and $v = u_{\nu^*} = \mathbb{K}_V[\nu^*]$ by (4.12). It is clear that $\nu^* \leqslant \nu$ and that the problem (4.13) admits a solution for $\lambda = \nu^*$. If λ is a positive measure smaller than μ, then $\lambda^* \leqslant \mu^*$. But if there exists some λ such that the problem (4.13) admits a solution, then $\lambda = \lambda^*$. This implies the claim. □

As a consequence of the characterization of ν^* there holds

Corollary 4.3. *Assume $V \geqslant 0$ and let $\{V_k\}$ be an increasing sequence of nonnegative bounded measurable functions converging to V a.e. in Ω. Then the solution u_k of*

$$\begin{cases} -\Delta u + V_k u = 0 & \textit{in } \Omega, \\ u = \nu & \textit{in } \partial\Omega, \end{cases} \tag{4.14}$$

converges to u_{ν^*}.

Proof. The previous construction shows that $u_k = \mathbb{K}_{V_k}[\nu]$ decreases to some $\tilde{u}$ which satisfies a relaxed equation, the boundary data of which, $\tilde{\nu}^*$, is the largest measure $\lambda \leqslant \nu$ for which problem (4.13) admits a solution. Therefore $\tilde{\nu}^* = \nu^*$ and $\tilde{u} = u_{\nu^*}$. Similarly $\{K_{V_k}^{\Omega}\}$ decreases and converges to K_V^{Ω}. □

We define the *boundary vanishing set of* K_V^{Ω} by

$$\mathcal{S}ing_V(\Omega) := \big\{ y \in \partial\Omega \;\big|\; K_V^{\Omega}(x, y) = 0 \big\} \quad \text{for some } x \in \Omega. \tag{4.15}$$

Since $V \in L^{\infty}_{loc}(\Omega)$, $\mathcal{S}ing_V(\Omega)$ is independent of x by Harnack inequality; furthermore it is a Borel set. This set is called the set of *finely irregular boundary points* by E.B. Dynkin; the reason for such a denomination will appear in Appendix A.

Theorem 4.4. *Let* $\nu \in \mathfrak{M}_+(\partial\Omega)$.

(i) *If* $\nu((\mathcal{S}ing_V(\Omega))^c) = 0$, *then* $\nu^* = 0$.
(ii) *There always holds* $\mathcal{S}ing_V(\Omega) \subset Z_V$.

Proof. The first assertion is clear since $\nu = \chi_{\mathcal{S}ing_V(\Omega)}\nu + \chi_{\mathcal{S}ing_V(\Omega)^c}\nu = \chi_{\mathcal{S}ing_V(\Omega)}\nu$ and, by Proposition 4.2,

$$u_{\nu^*}(x) = \mathbb{K}_V\big[\nu^*\big](x) = \int\limits_{\mathcal{S}ing_V(\Omega)} K_V^{\Omega}(x, y)\, d\nu(y) = 0, \quad \forall x \in \Omega,$$

by definition of $\mathcal{S}ing_V(\Omega)$. For proving (ii), we assume that $C_V(\mathcal{S}ing_V(\Omega)) > 0$; there exists $\mu \in \mathfrak{M}_+^V(\mathcal{S}ing_V(\Omega))$ such that $\mu(\mathcal{S}ing_V(\Omega)) > 0$. Since μ is admissible let u_μ be the solution of (1.1). Then $\mu^* = \mu$, thus $u_\mu = \mathbb{K}^V[\mu]$ and

$$\mathbb{K}^V[\mu](x) = \int\limits_{\partial\Omega} K_V^{\Omega}(x, y)\, d\mu(y) = \int\limits_{\mathcal{S}ing_V(\Omega)} K_V^{\Omega}(x, y)\, d\mu(y) = 0,$$

contradiction. Thus $C_V(\mathcal{S}ing_V(\Omega)) = 0$. Since (3.9) implies that Z_V is the largest Borel set with zero C_V-capacity, it implies $\mathcal{S}ing_V(\Omega) \subset Z_V$. □

In order to obtain more precise informations on $\mathcal{S}ing_V(\Omega)$ some minimal regularity assumptions on V are needed. We also recall the following result due to Ancona [5] and developed in Appendix A of the present work.

Theorem 4.5. *Assume* $V \geqslant 0$ *satisfies* $\delta_\Omega^2 V \in L^\infty(\Omega)$. *If for some* $y \in \partial\Omega$ *and some cone* C_y *with vertex* y *such that* $\overline{C}_y \cap B_r(y) \subset \Omega \cup \{y\}$ *for some* $r > 0$ *there holds*

$$\int\limits_{C_y} \frac{V(x)}{|x - y|^{N-2}}\, dx = \infty, \tag{4.16}$$

then

$$K_V^{\Omega}(x,y)=0, \quad \forall x\in\Omega. \tag{4.17}$$

This means that (4.16) implies that y belongs to $\mathcal{S}ing_V(\Omega)$. Set $\delta_\Omega(x)=\operatorname{dist}(x,\partial\Omega)$. We define the *conical singular boundary set*

$$\tilde{Z}_V=\left\{y\in\partial\Omega\colon \int\limits_{C_{\epsilon,y}} K^{\Omega}(x,y)V(x)\phi(x)\,dx=\infty \text{ for some } \epsilon>0\right\} \tag{4.18}$$

where $C_{\epsilon,y}:=\{x\in\Omega\colon \delta_\Omega(x)\geqslant\epsilon|x-y|\}$. Clearly $\tilde{Z}_V\subset Z_V$.

Corollary 4.6. *Assume $V\geqslant 0$ satisfies $\delta_\Omega^2 V\in L^\infty(\Omega)$. Then $\tilde{Z}_V\subset\mathcal{S}ing_V(\Omega)$.*

Proof. Let $y\in\tilde{Z}_V$. Since there exists $c>0$ such that

$$c^{-1}V(x)|x-y|^{2-N}\leqslant K^{\Omega}(x,y)V(x)\phi(x)\leqslant cV(x)|x-y|^{2-N}, \quad \forall x\in C_{\epsilon,y} \tag{4.19}$$

the result follows immediately from (4.16), (4.18). $\square$

Remark. In situations coming from the nonlinear equation $-\Delta u+|u|^{q-1}u=0$ in Ω with $q>1$, $V=|u|^{q-1}$ not only satisfies $gd_\Omega^2 V\in L^\infty(\Omega)$ but also the restricted oscillation condition: for any $y\in\partial\Omega$ and any open cone C_y with vertex y such that $C_y\Subset\Omega$, there exists $c>0$ such that

$$\forall(x,z)\in C_y\times C_y, \quad |x-y|=|z-y| \quad\Rightarrow\quad c^{-1}\leqslant\frac{V(x)}{V(z)}\leqslant c. \tag{4.20}$$

It is a consequence of the Keller–Osserman estimate and Harnack inequality. In this case condition (4.16) is equivalent to

$$\int_0^1 V\big(\gamma(t)\big)t\,dt=\infty, \tag{4.21}$$

at least for one path $\gamma\in C^{0,1}([0,1])$ such that $\gamma(0)=y$ and $\gamma((0,1])\subset C_y$ for some cone $C_y\Subset\Omega$.

5. The boundary trace

5.1. *The regular part*

In this section, $V\in L^\infty_{loc}(\Omega)$ is nonnegative. If $0<\epsilon\leqslant\epsilon_0$, we denote $\delta_\Omega(x)=\operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$, and set $\Omega_\epsilon:=\{x\in\Omega\colon \delta_\Omega(x)>\epsilon\}$, $\Omega'_\epsilon=\Omega\setminus\Omega_\epsilon$ and $\Sigma_\epsilon=\partial\Omega_\epsilon$. It is well known that there exists ϵ_0 such that, for any $0<\epsilon\leqslant\epsilon_0$ and any $x\in\Omega'_\epsilon$ there exists a unique projection $\sigma(x)$ of x on $\partial\Omega$ and any $x\in\Omega'_\epsilon$ can be written in a unique way under the form

$$x = \sigma(x) - \delta_\Omega(x)\mathbf{n}$$

where $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$ at $\sigma(x)$. The mapping $x \mapsto (\delta_\Omega(x), \sigma(x))$ is a C^2 diffeomorphism from Ω'_ϵ to $(0, \epsilon_0] \times \partial\Omega$. We recall the following definition given in [20]. If $\mathcal{A}$ is a Borel subset of $\partial\Omega$, we set $\mathcal{A}_\epsilon = \{x \in \Sigma_\epsilon\colon \sigma(x) \in A\}$.

Definition 5.1. Let $\mathcal{A}$ be a relatively open subset of $\partial\Omega$, $\{\mu_\epsilon\}$ be a set of Radon measures on $\mathcal{A}_\epsilon$ $(0 < \epsilon \leqslant \epsilon_0)$ and $\mu \in \mathfrak{M}(\mathcal{A})$. We say that $\mu_\epsilon \rightharpoonup \mu$ in the weak*-topology if, for any $\zeta \in C_c(\mathcal{A})$,

$$\lim_{\epsilon\to 0} \int_{\mathcal{A}_\epsilon} \zeta\big(\sigma(x)\big)\, d\mu_\epsilon(x) = \int_{\mathcal{A}} \zeta\, d\mu. \tag{5.1}$$

A function $u \in C(\Omega)$ possesses a boundary trace $\mu \in \mathfrak{M}(\mathcal{A})$ if

$$\lim_{\epsilon\to 0} \int_{\mathcal{A}_\epsilon} \zeta\big(\sigma(x)\big)u(x)\, dS(x) = \int_{\mathcal{A}} \zeta\, d\mu, \quad \forall\zeta \in C_c(\mathcal{A}). \tag{5.2}$$

The following result is proved in [20, p. 694].

Proposition 5.2. *Let $u \in C(\Omega)$ be a positive solution of*

$$-\Delta u + V(x)u = 0 \quad \text{in } \Omega. \tag{5.3}$$

Assume that, for some $z \in \partial\Omega$, there exists an open neighborhood U of z such that

$$\int_{U\cap\Omega} V u\phi(x)\, dx < \infty. \tag{5.4}$$

Then $u \in L^1(K \cap \Omega)$ for any compact subset $K \subset G$ and there exists a positive Radon measure μ on $\mathcal{A} = U \cap \partial\Omega$ such that

$$\lim_{\epsilon\to 0} \int_{U\cap\Sigma_\epsilon} \zeta\big(\sigma(x)\big)u(x)\, dS(x) = \int_{\mathcal{A}} \zeta\, d\mu, \quad \forall\zeta \in C_c(U \cap \Omega). \tag{5.5}$$

Notice that any continuous solution of (5.3) in Ω belongs to $W^{2,p}_{loc}(\Omega)$ for any $(1 \leqslant p < \infty)$. This previous result yields to a natural definition of the regular boundary points.

Definition 5.3. Let $u \in C(\Omega)$ be a positive solution of (5.3). A point $z \in \partial\Omega$ is called a regular boundary point for u if there exists an open neighborhood U of z such that (5.5) holds. The set of regular boundary points is a relatively open subset of $\partial\Omega$, denoted by $\mathcal{R}(u)$. The set $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is the singular boundary set of u. It is a closed set.

By Proposition 5.2 and using a partition of unity, we see that there exists a positive Radon measure $\mu := \mu_u$ on $\mathcal{R}(u)$ such that (5.5) holds with U replaced by $\mathcal{R}(u)$. The couple $(\mu_u, \mathcal{S}(u))$

is called the **boundary trace of** u. *The main question of the boundary trace problem is to analyze the behaviour of u near the set $\mathcal{S}(u)$.*

For any positive good measure μ on $\partial\Omega$, we denote by u_μ the solution of (4.1) defined by (4.10)–(4.11).

Proposition 5.4. *Let $u \in C(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for any $(1 \leqslant p < \infty)$ be a positive solution of* (5.3) *in Ω with boundary trace $(\mu_u, \mathcal{S}(u))$ such that μ_u is bounded. Then μ_u is good and $u \geqslant u_{\mu_u}$. Then $u \geqslant u_{\mu_u}$.*

Proof. Let $G \subset \partial\Omega$ be a relatively open subset such that $\overline{G} \subset \mathcal{R}(u)$ with a C^2 relative boundary $\partial^* G = \overline{G} \setminus G$. There exists an increasing sequence of C^2 domains Ω_n such that $\overline{G} \subset \partial\Omega_n$, $\partial\Omega_n \setminus \overline{G} \subset \Omega$ and $\bigcup_n \Omega_n = \Omega$. For any n, let $v := v_n$ be the solution of

$$\begin{cases} -\Delta v + Vv = 0 & \text{in } \Omega_n, \\ v = \chi_G \mu & \text{in } \partial\Omega_n. \end{cases} \tag{5.6}$$

Let u_n be the restriction of u to Ω_n. Since $u \in C(\Omega)$ and $Vu\phi \in L^1(\Omega_n)$, there also holds $Vu\phi_n \in L^1(\Omega_n)$ where we have denoted by ϕ_n the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega_n)$. Consequently u_n admits a regular boundary trace μ_n on $\partial\Omega_n$ (i.e. $\mathcal{R}(u_n) = \partial\Omega_n$) and u_n is the solution of

$$\begin{cases} -\Delta v + Vv = 0 & \text{in } \Omega_n, \\ v = \mu_n & \text{in } \partial\Omega_n. \end{cases} \tag{5.7}$$

Furthermore $\mu_n|_G = \chi_G \mu_u$. It follows from Brezis estimates and in particular (2.5) that $u_n \leqslant u$ in Ω_n. Since $\Omega_n \subset \Omega_{n+1}$, $v_n \leqslant v_{n+1}$. Moreover

$$v_n + \mathbb{G}^{\Omega_n}[Vv_n] = \mathbb{K}^{\Omega_n}[\chi_G \mu] \quad \text{in } \Omega_n.$$

Since $\mathbb{K}^{\Omega_n}[\chi_G \mu_u] \to \mathbb{K}^{\Omega}[\chi_G \mu_u]$, and the Green kernels $G^{\Omega_n}(x, y)$ are increasing with n, it follows from monotone convergence that $v_n \uparrow v$ and there holds

$$v + \mathbb{G}^{\Omega}[Vv] = \mathbb{K}^{\Omega}[\chi_G \mu_u] \quad \text{in } \Omega.$$

Thus $v = u_{\chi_G \mu_u}$ and $u_{\chi_G \mu_u} \leqslant u$. We can now replace G by a sequence $\{G_k\}$ of relatively open sets with the same properties as G, $\overline{G}_k \subset G_k$ and $\bigcup_k G_k = \mathcal{R}(u)$. Then $\{u_{\chi_{G_k} \mu_u}\}$ is increasing and converges to some $\tilde{u}$. Since

$$u_{\chi_{G_k} \mu_u} + \mathbb{G}^{\Omega}[Vu_{\chi_{G_k} \mu_u}] = \mathbb{K}^{\Omega}[\chi_{G_k} \mu_u],$$

and $\mathbb{K}^{\Omega}[\chi_{G_k} \mu] \uparrow \mathbb{K}^{\Omega}[\mu_u]$, we derive

$$\tilde{u} + \mathbb{G}^{\Omega}[V\tilde{u}] = \mathbb{K}^{\Omega}[\mu_u].$$

This implies that $\tilde{u} = u_{\mu_u} \leqslant u$. $\square$

Remark. If μ_u is not bounded, there still holds $\tilde{u} \leqslant u$, but the singular set of the boundary trace of $\tilde{u}$ is not empty.

5.2. The singular part

The following result is essentially proved in [20, Lemma 2.8].

Proposition 5.5. *Let $u \in C(\Omega)$ for any $(1 \leqslant p < \infty)$ be a positive solution of (5.3) and suppose that $z \in \mathcal{S}(u)$ and that there exists an open neighborhood U_0 of z such that $u \in L^1(\Omega \cap U_0)$. Then for any open neighborhood U of z, there holds*

$$\lim_{\epsilon \to 0} \int_{U \cap \Sigma_\epsilon} \zeta\big(\sigma(x)\big) u(x)\, dS(x) = \infty. \tag{5.8}$$

As immediate consequences, we have

Corollary 5.6. *Assume u satisfies the regularity assumption of Proposition 5.4. Then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z, there holds*

$$\limsup_{\epsilon \to 0} \int_{U \cap \Sigma_\epsilon} \zeta\big(\sigma(x)\big) u(x)\, dS(x) = \infty. \tag{5.9}$$

Corollary 5.7. *Assume u satisfies the regularity assumption of Proposition 5.4. If $u \in L^1(\Omega)$, then for any $z \in \mathcal{S}(u)$ and any open neighborhood U of z, (5.8) holds.*

The two next results give conditions on V which imply that $\mathcal{S}(u) = \emptyset$.

Theorem 5.8. *Assume $N = 2$, V is nonnegative and satisfies (2.19). If u is a positive solution of (5.3), then $\mathcal{R}(u) = \partial\Omega$.*

Proof. We assume that

$$\int_\Omega V \phi u\, dx = \infty. \tag{5.10}$$

If $0 < \epsilon \leqslant \epsilon_0$, we denote by $(\phi_\epsilon, \lambda_\epsilon)$ are the normalized first eigenfunction and first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega_\epsilon)$, then

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} V \phi_\epsilon u\, dx = \infty. \tag{5.11}$$

Because

$$\int_{\Omega_\epsilon} (\lambda_\epsilon + \phi_\epsilon V) u\, dx = -\int_{\partial\Omega_\epsilon} \frac{\partial \phi_\epsilon}{\partial \mathbf{n}} u\, dS,$$

and

$$c^{-1} \leqslant -\frac{\partial \phi_\epsilon}{\partial \mathbf{n}} \leqslant c,$$

for some $c > 1$ independent of ϵ, there holds

$$\lim_{\epsilon \to 0} \int_{\partial \Omega_\epsilon} u \, dS = \infty. \tag{5.12}$$

Denote by m_ϵ this last integral and set $v_\epsilon = m_\epsilon^{-1} u$ and $\mu_\epsilon = m_\epsilon^{-1} u|_{\partial \Omega_\epsilon}$. Then

$$v_\epsilon + \mathbb{G}^{\Omega_\epsilon}[V v_\epsilon] = \mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon] \quad \text{in } \Omega_\epsilon \tag{5.13}$$

where

$$\mathbb{K}^{\Omega_\epsilon}[\mu_\epsilon](x) = \int_{\partial \Omega_\epsilon} K^{\Omega_\epsilon}(x, y) \mu_\epsilon(y) \, dS(y) \tag{5.14}$$

is the Poisson potential of μ_ϵ in Ω_ϵ and

$$\mathbb{G}^{\Omega_\epsilon}[V u](x) = \int_{\Omega_\epsilon} G^{\Omega_\epsilon}(x, y) V(y) u(y) \, dy,$$

the Green potential of Vu in Ω_ϵ. Furthermore

$$\begin{cases} -\Delta v_\epsilon + V v_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ v_\epsilon = \mu_\epsilon & \text{in } \partial \Omega_\epsilon. \end{cases} \tag{5.15}$$

By Brezis estimates and regularity theory for elliptic equations, $\{\chi_{\Omega_\epsilon} v_\epsilon\}$ is relatively compact in $L^1(\Omega)$ and in the local uniform topology of Ω_ϵ. Up to a subsequence $\{\epsilon_n\}$, μ_{ϵ_n} converges to a probability measure μ on $\partial \Omega$ in the weak*-topology. It is classical that

$$\mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}] \to \mathbb{K}[\mu]$$

locally uniformly in Ω, and $\chi_{\Omega_{\epsilon_n}} v_{\epsilon_n} \to v$ in the local uniform topology of Ω, and a.e. in Ω. Because $G^{\Omega_\epsilon}(x, y) \uparrow G^{\Omega}(x, y)$, there holds for any $x \in \Omega$

$$\lim_{n \to \infty} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x, y) V(y) v_{\epsilon_n}(y) = G^{\Omega}(x, y) V(y) v(y) \quad \text{for almost all } y \in \Omega. \tag{5.16}$$

Furthermore $v_{\epsilon_n} \leqslant \mathbb{K}^{\Omega_{\epsilon_n}}[\mu_{\epsilon_n}]$ reads

$$v_{\epsilon_n}(y) \leqslant c \phi_{\epsilon_n}(y) \int_{\partial \Omega_n} \frac{\mu_{\epsilon_n}(z) \, dS(z)}{|y - z|^2}.$$

In order to go to the limit in the expression

$$L_n := \mathbb{G}^{\Omega_{\epsilon_n}}[V v_{\epsilon_n}](x) = \int_{\Omega} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy, \tag{5.17}$$

we may assume that $x \in \Omega_{\epsilon_1}$ where $0 < \epsilon_1 \leqslant \epsilon_0$ is fixed and write $\Omega = \Omega_{\epsilon_1} \cup \Omega'_{\epsilon_1}$ where

$$\Omega'_{\epsilon_1} = \Omega \setminus \Omega_{\epsilon_1} := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \leqslant \epsilon_1\}$$

and $L_n = M_n + P_n$ where

$$M_n = \int_{\Omega_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy \tag{5.18}$$

and

$$P_n = \int_{\Omega'_{\epsilon_1}} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy. \tag{5.19}$$

Since

$$\begin{aligned}
\chi_{\Omega_{\epsilon_1}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y) &\leqslant c\chi_{\Omega_{\epsilon_1}}(y)\big|\ln\big(|x-y|\big)\big| V(y) v_{\epsilon_n}(y) \\
&\leqslant c\|V\|_{L^\infty(\Omega_{\epsilon_1})} \chi_{\Omega_{\epsilon_1}}(y)\big|\ln\big(|x-y|\big)\big| v_{\epsilon_n}(y),
\end{aligned}$$

it follows by the dominated convergence theorem that

$$\lim_{n\to\infty} M_n = \int_{\Omega_{\epsilon_1}} G^{\Omega}(x,y) V(y) v(y)\,dy. \tag{5.20}$$

Let $E \subset \Omega$ be a Borel subset. Then $G^{\Omega_{\epsilon_n}}(x,y) \leqslant c(x)\phi_{\epsilon_n}(y)$ if $y \in \Omega'_{\epsilon_1}$. By Fubini,

$$\begin{aligned}
&\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy \\
&\quad\leqslant cc(x) \int_{\partial\Omega_n} \left(\int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\phi^2_{\epsilon_n}(y) V(y)}{|y-z|^2}\,dy \right) \mu_{\epsilon_n}(z)\,dS(z) \\
&\quad\leqslant cc(x) \max_{z \in \partial\Omega_{\epsilon_n}} \int_{\Omega'_{\epsilon_1} \cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\phi^2_{\epsilon_n}(y) V(y)}{|y-z|^2}\,dy.
\end{aligned} \tag{5.21}$$

If $y \in \Omega_{\epsilon_n} \cap E$, there holds $\phi(y) = \phi_{\epsilon_n}(y) + \epsilon_n$. If $z \in \partial\Omega_{\epsilon_n} \cap E$ and we denote by $\sigma(z)$ the projection of z onto $\partial\Omega$, there holds $|y - \sigma(z)| \leqslant |y - z| + \epsilon_n$. By monotonicity

$$\frac{\phi_{\epsilon_n}(y)}{|y-z|} \leqslant \frac{\phi_{\epsilon_n}(y) + \epsilon_n}{|y-z| + \epsilon_n} \leqslant \frac{\phi(y)}{|y - \sigma(z)|}, \tag{5.22}$$

thus

$$\int\limits_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy$$
$$\leqslant cc(x) \max_{z\in\partial\Omega} \int\limits_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\phi^2(y)V(y)}{|y-z|^2}\,dy. \tag{5.23}$$

By (2.19) this last integral goes to zero if $|\Omega'_{\epsilon_1}\cap E\cap\Omega_{\epsilon_n}|\to 0$. Thus by Vitali's theorem, the sequence of functions $\{\chi_{\Omega_{\epsilon_n}}(.)G^{\Omega_{\epsilon_n}}(x,.)V(y)v_{\epsilon_n}(.)\}_{n\in\mathbb{N}}$ is uniformly integrable in y, for any $x\in\Omega$. It implies that

$$\lim_{n\to\infty}\int\limits_{\Omega} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy = \int\limits_{\Omega} G^{\Omega}(x,y)V(y)v(y)\,dy, \tag{5.24}$$

and there holds $v+\mathbb{G}[Vv]=\mathbb{K}[\mu]$. Since $u=m_\epsilon v_\epsilon$ in Ω and $m_\epsilon\to\infty$, we get a contradiction since it would imply $u\equiv\infty$. □

In order to deal with the case $N\geqslant 3$ we introduce an additional assumption of stability.

Theorem 5.9. *Assume $N\geqslant 3$. Let $V\in L^\infty_{loc}(\Omega)$, $V\geqslant 0$ such that*

$$\lim_{\substack{E \text{ Borel}\\ |E|\to 0}} \int\limits_E V(y)\frac{(\phi(y)-\epsilon)_+^2}{|y-z|^N}\,dy = 0 \quad \textit{uniformly with respect to } z\in\Sigma_\epsilon \textit{ and } \epsilon\in(0,\epsilon_0]. \tag{5.25}$$

If u is a positive solution of (5.3)*, then $\mathcal{R}(u)=\partial\Omega$.*

Proof. We proceed as in Theorem 5.8. All the relations (5.10)–(5.20) are valid and (5.21) has to be replaced by

$$\int\limits_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy$$
$$\leqslant cc(x) \max_{z\in\Sigma_{\epsilon_n}} \int\limits_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) \frac{\phi^2_{\epsilon_n}(y)V(y)}{|y-z|^{N+1}}\,dy. \tag{5.26}$$

Since (5.22) is no longer valid, (5.22) is replaced by

$$\int_{\Omega'_{\epsilon_1}\cap E} \chi_{\Omega_{\epsilon_n}}(y) G^{\Omega_{\epsilon_n}}(x,y) V(y) v_{\epsilon_n}(y)\,dy$$
$$\leqslant cc(x) \max_{z\in\Sigma_{\epsilon_n}} \int_E V(y) \frac{(\phi(y)-\epsilon_n)_+^2}{|y-z|^{N+1}}\,dy. \tag{5.27}$$

By (5.25) the left-hand side of (5.27) goes to zero when $|E|\to 0$, uniformly with respect to ϵ_n. This implies that (5.24) is still valid and the conclusion of the proof is as in Theorem 5.8. □

Remark. A simpler statement which implies (5.25) is the following

$$\lim_{\delta\to 0} \int_0^{\delta} \left(\int_{B_r(z)} V(y) \big(\phi(y)-\epsilon\big)_+^2\,dy \right) \frac{dr}{r^{N+1}} = 0, \tag{5.28}$$

uniformly with respect to $0<\epsilon\leqslant\epsilon_0$ and to $z\in\Sigma_\epsilon$. The proof is similar to the one of Proposition 2.7.

Remark. When the function V depends essentially of the distance to $\partial\Omega$ in the sense that

$$\big|V(x)\big| \leqslant v\big(\phi(x)\big), \quad \forall x\in\Omega, \tag{5.29}$$

and v satisfies

$$\int_0^a t v(t)\,dt < \infty, \tag{5.30}$$

Marcus and Véron proved [20, Lemma 7.4] that $\mathcal{R}(u)=\partial\Omega$, for any positive solution u of (5.3). This assumption implies also (5.25). The proof is similar to the one of Proposition 2.8.

5.3. The sweeping method

This method introduced in [25] for analyzing isolated singularities of solutions of semilinear equations has been adapted in [19] and [21] for defining an extended trace of positive solutions of differential inequalities in particular in the super-critical case. Since the boundary trace of a positive solutions of (5.3) is known on $\mathcal{R}(u)$ we shall study the sweeping with measure concentrated on the singular set $\mathcal{S}(u)$.

Proposition 5.10. *Let $u\in C(\Omega)$ be a positive solution of* (5.3) *with singular boundary set $\mathcal{S}(u)$. If $\mu\in\mathfrak{M}_+(\mathcal{S}(u))$ we denote $v_\mu=\inf\{u,u_\mu\}$. Then*

$$-\Delta v_\mu + V(x) v_\mu \geqslant 0 \quad \text{in } \Omega, \tag{5.31}$$

and v_μ admits a boundary trace $\gamma_u(\mu)\in\mathfrak{M}_+(\mathcal{S}(u))$. The mapping $\mu\mapsto\gamma_u(\mu)$ is nondecreasing and $\gamma_u(\mu)\leqslant\mu$.

Proof. By [26], (5.31) holds. But $Vu_\mu \in L^1_\phi(\Omega) \Rightarrow Vv_\mu \in L^1_\phi(\Omega)$, if we set $w := \mathbb{G}[Vv_\mu]$, then $v_\mu + w$ is nonnegative and super-harmonic, thus it admits a boundary trace in $\mathfrak{M}_+(\partial\Omega)$ that we denote by $\gamma_u(\mu)$. Clearly $\gamma_u(\mu) \leqslant \mu$ since $v_\mu \leqslant u_\mu$ and $\gamma_u(\mu)$ is nondecreasing with μ as $\mu \mapsto u_\mu$ is. Finally, since v_μ is a supersolution, it is larger that the solution of (5.3) with the same boundary trace $\gamma_u(\mu)$, and there holds

$$u_{\gamma_u(\mu)} \leqslant v_\mu. \qquad \square \tag{5.32}$$

Proposition 5.11. *Let*

$$\nu_S(u) := \sup\bigl\{\gamma_u(\mu)\colon\ \mu \in \mathfrak{M}_+\bigl(\mathcal{S}(u)\bigr)\bigr\}. \tag{5.33}$$

Then $\nu_S(u)$ is a Borel measure on $\mathcal{S}(u)$.

Proof. We borrow the proof to Marcus and Véron [21], and we naturally extend any positive Radon measure to a positive bounded and regular Borel measure by using the same notation. It is clear that $\nu_S(u) := \nu_S$ is an outer measure in the sense that

$$\nu_S(\emptyset) = 0, \quad \text{and} \quad \nu_S(A) \leqslant \sum_{k=1}^{\infty} \nu(A_k), \quad \text{whenever } A \subset \bigcup_{k=1}^{\infty} A_k. \tag{5.34}$$

Let A and $B \subset \mathcal{S}(u)$ be disjoint Borel subsets. In order to prove that

$$\nu_S(A \cup B) = \nu_S(A) + \nu_S(B), \tag{5.35}$$

we first notice that the relation holds if $\max\{\nu_S(A), \nu_S(B)\} = \infty$. Therefore we assume that $\nu_S(A)$ and $\nu_S(B)$ are finite. For $\varepsilon > 0$ there exist two bounded positive measures μ_1 and μ_2 such that

$$\gamma_u(\mu_1)(A) \leqslant \nu(A) \leqslant \gamma_u(\mu_1)(A) + \varepsilon/2$$

and

$$\gamma_u(\mu_2)(B) \leqslant \nu(B) \leqslant \gamma_u(\mu_2)(B) + \varepsilon/2.$$

Hence

$$\begin{aligned}
\nu_S(A) + \nu_S(B) &\leqslant \gamma_u(\mu_1)(A) + \gamma_u(\mu_2)(B) + \varepsilon \\
&\leqslant \gamma_u(\mu_1 + \mu_2)(A) + \gamma_u(\mu_1 + \mu_2)(B) + \varepsilon \\
&= \gamma_u(\mu_1 + \mu_2)(A \cup B) + \varepsilon \\
&\leqslant \nu_S(A \cup B) + \varepsilon.
\end{aligned}$$

Therefore ν_S is a finitely additive measure. If $\{A_k\}$ ($k \in \mathbb{N}$) is a sequence of disjoint Borel sets and $A = \bigcup A_k$, then

$$\nu_S(A) \geqslant \nu_S\Bigl(\bigcup_{1\leqslant k\leqslant n} A_k\Bigr) = \sum_{k=1}^{n} \nu_S(A_k) \quad \Rightarrow \quad \nu_S(A) \geqslant \sum_{k=1}^{\infty} \nu_S(A_k).$$

By (5.34), it implies that ν_S is a countably additive measure. □

Definition 5.12. The Borel measure $\nu(u)$ defined by

$$\nu(u)(A) := \nu_S\bigl(A \cap \mathcal{S}(u)\bigr) + \mu_u\bigl(A \cap \mathcal{R}(u)\bigr), \quad \forall A \subset \partial\Omega,\ A \text{ Borel}, \tag{5.36}$$

is called the extended boundary trace of u, denoted by $Tr^e(u)$.

Proposition 5.13. *If $A \subset \mathcal{S}(u)$ is a Borel set, then*

$$\nu_S(A) := \sup\bigl\{\gamma_u(\mu)(A):\ \mu \in \mathfrak{M}_+(A)\bigr\}. \tag{5.37}$$

Proof. If $\lambda, \lambda' \in \mathfrak{M}_+(\mathcal{S}(u))$

$$\inf\{u, u_{\lambda+\lambda'}\} = \inf\{u, u_\lambda + u_{\lambda'}\} \leqslant \inf\{u, u_\lambda\} + \inf\{u, u_{\lambda'}\}.$$

Since the three above functions admit a boundary trace, it follows that

$$\gamma_u\bigl(\lambda + \lambda'\bigr) \leqslant \gamma_u(\lambda) + \gamma_u\bigl(\lambda'\bigr).$$

If A is a Borel subset of $\mathcal{S}(u)$, then $\mu = \mu_A + \mu_{A^c}$ where $\mu_A = \chi_E \mu$. Thus

$$\gamma_u(\mu) \leqslant \gamma_u(\mu_A) + \gamma_u(\mu_{A^c}),$$

and

$$\gamma_u(\mu)(A) \leqslant \gamma_u(\mu_A)(A) + \gamma_u(\mu_{A^c})(A).$$

Since $\gamma_u(\mu_{A^c}) \leqslant \mu_{A^c}$ and $\mu_{A^c}(A) = 0$, it follows

$$\gamma_u(\mu)(A) \leqslant \gamma_u(\mu_A)(A).$$

But $\mu_A \leqslant \mu$, thus $\gamma_u(\mu_A) \leqslant \gamma_u(\mu)$ and finally

$$\gamma_u(\mu)(A) = \gamma_u(\mu_A)(A). \tag{5.38}$$

If $\mu \in \mathfrak{M}_+(A)$, $\mu = \mu_A$, thus (5.37) follows. □

Proposition 5.14. *There always holds*

$$\nu(u)\big(\mathcal{S}ing_V(\Omega)\big) = 0, \tag{5.39}$$

where $\mathcal{S}ing_V(\Omega)$ is the vanishing set of $K_V^\Omega(x,.)$ defined by (4.15).

Proof. This follows from the fact that for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ concentrated on $\mathcal{S}ing_V(\Omega)$, $u_\mu = 0$. Thus $\gamma_u(\mu) = 0$. If μ is a general measure, we can write $\mu = \chi_{\mathcal{S}ing_V(\Omega)}\mu + \chi_{(\mathcal{S}ing_V(\Omega))^c}\mu$, thus $u_\mu = u_{\chi_{(\mathcal{S}ing_V(\Omega))^c}\mu}$. Because of (5.32)

$$\gamma_u(\mu)\big(\mathcal{S}ing_V(\Omega)\big) = \gamma_u(\chi_{(\mathcal{S}ing_V(\Omega))^c}\mu)\big(\mathcal{S}ing_V(\Omega)\big) \leqslant (\chi_{(\mathcal{S}ing_V(\Omega))^c}\mu)\big(\mathcal{S}ing_V(\Omega)\big) = 0,$$

thus (5.39) holds. □

Remark. This process for determining the boundary trace is ineffective if there exist positive solutions u in Ω such that

$$\lim_{\delta_\Omega(x)\to 0} u(x) = \infty.$$

This is the case if $\Omega = B_R$ and $V(x) = c(R-|x|)^{-2}$ $(c > 0)$. In this case $K_V^\Omega(x,.) \equiv 0$. For any $a > 0$, there exists a radial solution of

$$-\Delta u + \frac{cu}{(R-|x|)^2} = 0 \quad \text{in } B_R \tag{5.40}$$

under the form

$$u(r) = u_a(r) = a + c\int_0^r s^{1-N}\int_0^s u(t)\frac{t^{N-1}\,dt}{(R-t)^2}. \tag{5.41}$$

Such a solution is easily obtained by fixed point, $u(0) = a$ and the above formula shows that u_a blows up when $r \uparrow R$. We do not know if there exist non-radial positive solutions of (5.40). More generally, if Ω is a smooth bounded domain, we do not know if there exists a non-trivial positive solution of

$$-\Delta u + \frac{c}{d^2(x)}u = 0 \quad \text{in } \Omega. \tag{5.42}$$

Theorem 5.15. *Assume $V \geqslant 0$ and satisfies* (2.19)*. If u is a positive solution of* (5.3)*, then $Tr^e(u) = \nu(u)$ is a bounded measure.*

Proof. Set $\nu = \nu(u)$ and assume $\nu(\partial\Omega) = \infty$. By dichotomy there exists a decreasing sequence of relatively open domains $D_n \subset \partial\Omega$ such that $\overline{D}_n \subset D_{n-1}$, diam $D_n = r_n \to 0$ as $n \to \infty$, and $\nu(D_n) = \infty$. For each n, there exists a Radon measure $\mu_n \in \mathfrak{M}_+(D_n)$ such that $\gamma_u(\mu_n)(D_n) = n$, and

$$u \geqslant v_{\mu_n} = \inf\{u, u_{\mu_n}\} \geqslant u_{\gamma_u(\mu_n)}.$$

Set $m_n = n^{-1}\gamma_u(\mu_n)$, then $m_n \in \mathfrak{M}_+(D_n)$ has total mass 1 and it converges in the weak*-topology to δ_a, where $\{a\} = \bigcap_n D_n$. By Theorem 2.6, u_{m_n} converges to u_{δ_a}. Since $u \geqslant nu_{m_n}$, it follows that

$$u \geqslant \lim_{n\to\infty} nu_{m_n} = \infty,$$

a contradiction. Thus ν is a bounded Borel measure (and thus outer regular) and it corresponds to a unique Radon measure. □

Remark. If $N = 2$, it follows from Theorem 5.8 that $u = u_\nu$ and thus the extended boundary trace coincides with the usual boundary trace. The same property holds if $N \geqslant 3$, if (5.25) holds.

Appendix A. A necessary condition for the fine regularity of a boundary point with respect to a Schrödinger equation

by Alano Ancona[1]

This appendix is devoted to the derivation of a sufficient condition – stated in Theorem A.1 below (Section A1) – for the *fine singularity* of a boundary point of a Lipschitz domain with respect to a potential V. This theorem answers a question communicated by Moshe Marcus and Laurent Véron to the author – and related to the work [22,23] by Marcus and Véron. See e.g. Theorem 4.5, part (a), in [22]. The expounded proof goes back to the unpublished manuscript [5]. In a forthcoming paper other criterions for fine regularity will be given – in particular a simple explicit necessary and sufficient condition for the fine regularity of a boundary point and a criteria for having almost everywhere regularity in a subset of the boundary.

The exposition can be read independently of the above paper of L. Véron and C. Yarur. The few notions necessary to the statement of Theorem A.1 are recalled in Section A1. Section A2 is devoted to some known basic preliminary results and the proof of Theorem A.1 is given in Section A3.

A.1. Framework, notations and main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Denote $\delta_\Omega(x) := d(x; \mathbb{R}^N \setminus \Omega)$ the distance from x to the complement of Ω in $\mathbb{R}^N$ and for $a > 0$, let $\mathcal{V}(\Omega, a)$ denote the set of all nonnegative measurable function $V : \Omega \to \mathbb{R}$ such that $V(x) \leqslant a/(\delta_\Omega(x))^2$ in Ω. We also let x_0 to denote a fixed reference point in Ω.

For $V \in \mathcal{V}(a, \Omega)$, we will consider the Schrödinger operator $L_V := \Delta - V$ associated with the potential V. Here Δ is the classical Laplacian in $\mathbb{R}^N$.

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The kernels K_y, $\tilde{K}_y^V$ and K_y^V. It is well known [16,17] that to each point $y \in \partial\Omega$ corresponds a unique positive harmonic function K_y in Ω that vanishes on $\partial\Omega$ and satisfies the normalization condition $K_y(x_0) = 1$. This function is the Martin kernel w.r. to the Laplacian in Ω with pole at y and normalized at x_0. It may also be seen as a Poisson kernel with respect to Δ in Ω.

The function K_y is obviously superharmonic in Ω with respect to L_V and we may hence consider its greatest L_V-harmonic minorant $\tilde{K}_y^V$ in Ω defining hence another kernel function at y.

By the results in [3] (see Section A2 below) it is also known that for each $y \in \partial\Omega$ there exists a unique positive L_V-harmonic function K_y^V in Ω that vanishes on $\partial\Omega \setminus \{y\}$ and satisfies $K_y^V(x_0) = 1$. Thus $\tilde{K}_y^V = c_y K_y^V$ with $c_y = \tilde{K}_y^V(x_0)$. Here a function $u : \Omega \to \mathbb{R}$ is L_V-harmonic if u is the continuous representative of a weak solution u of $L_V(u) = 0$ (so $u \in H^1_{loc}(\Omega)$ by assumption and necessarily $u \in W^{2,p}_{loc}(\Omega)$ for all $p < \infty$).

The set of "*finely*" regular boundary points with respect to L_V in Ω is

$$\mathcal{R}eg_V(\Omega) := \{y \in \partial\Omega;\ \tilde{K}_y^V > 0\} = \{y \in \partial\Omega;\ c_y > 0\} \tag{A.1}$$

– since c is u.s.c. this is a K_σ subset of $\partial\Omega$ – and the set of "*finely*" irregular boundary points is $\mathcal{S}ing_V(\Omega) := \partial\Omega \setminus \mathcal{R}eg_V(\Omega)$. These notions were introduced by E.B. Dynkin in his study of positive solutions in Ω of a nonlinear equation such as $\Delta u = u^q$, $q > 1$ – in which case, given u, we recover Dynkin's definition on taking $V = |u|^{q-1}$. See the books [11,12] of E.B. Dynkin and the references therein. From the probabilistic point of view, a boundary point $y \in \partial\Omega$ is L_V finely regular iff for the Brownian motion $\{\xi_s\}_{0 \leqslant s < \tau}$ starting say at x_0 and conditioned to exit from Ω at y, it holds that $\int_0^\tau V(\xi_s)\,ds < +\infty$ a.s., or in other words, iff the probability for this process to reach y when killed at the rate V is strictly positive.

Let us now state Theorem A.1. It answers the question (2005) of Marcus and Véron alluded to above: suppose that for sufficiently many Lipschitz path (resp. every linear path) $\gamma : [0, \eta] \to \overline{\Omega}$ such that $\gamma(0) = y$ and $d(\gamma(t), \partial\Omega) \geqslant c|\gamma(t) - y|$ for $0 \leqslant t \leqslant \eta$ and some $c > 0$, it holds that $\int_0^\eta t V(\gamma(t))\,dt = +\infty$, does it follow that y is finely singular w.r. to V and Ω?

Theorem A.1. *Let $y \in \partial\Omega$ and let $C_{\epsilon,y} := \{x \in \Omega;\ \delta_\Omega(x) \geqslant \varepsilon d(x, y)\}$ for $0 < \varepsilon < 1$. If*

$$\int_{C_{\epsilon,y}} V(x) \frac{dx}{|x - y|^{N-2}} = +\infty \tag{A.2}$$

for some $\varepsilon > 0$, then $y \in \mathcal{S}ing_V(\Omega)$.

A.2. Boundary Harnack principle for L_V

To prove Theorem A.1 we will rely on the main result of [3] (see also [4]) in well-known forms more or less explicit in [3] (see e.g. Theorem 5′ and Corollary 27 there) or [4]. In this section we state these needed ancillary results and fix some notations to be used in what follows.

Fix positive reals $r, \rho > 0$ such that $0 < 10r < \rho$ and let f be a $\frac{\rho}{10r}$ Lipschitz function in the ball $B_{N-1}(0, r)$ of $\mathbb{R}^{N-1}$ such that $f(0) = 0$ – we let $B_{N-1}(m, s)$ to denote the ball in $\mathbb{R}^{N-1}$ of center m and radius r. Define then the region $U_f(r,\rho)$ in $\mathbb{R}^N$ as follows

$$U_f(r,\rho) := \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \simeq \mathbb{R}^N;\ |x'| < r,\ f(x') < x_N < \rho\}. \tag{A.3}$$

We will also denote it U (leaving f, r and ρ implicit) when convenient. Set $\partial_\# U := \partial U \cap \{x = (x', x_N) \in \mathbb{R}^N;\ |x'| \leqslant r,\ x_N = f(x')\}$ and define $T(t) := B_{N-1}(0; tr) \times (-t\rho, +t\rho)$.

Recall $\mathcal{V}_a(U)$ is the set of all Borel nonnegative functions V in U such that $V(x) \leqslant \frac{a}{\delta(x)^2}$ for $x \in U$. For V Hölder continuous in $\overline{\Omega}$ (in fact for a natural class of second order elliptic operators) the following statement goes back to [2]. See also [9] for $V = 0$.

Lemma A.2. *Let $V \in \mathcal{V}_a(U)$ and set $L_V := \Delta - V$. There is a constant C depending only on N, a and $\frac{\rho}{r}$ such that for any two positive L_V-harmonic functions u and v in U that vanish on $\partial_\# U$,*

$$\frac{u(x)}{u(A)} \leqslant C \frac{v(x)}{v(A)} \quad \text{for all } x \in U \cap T\left(\frac{1}{2}\right) \tag{A.4}$$

where $A = A_U = (0, \ldots, 0, \frac{\rho}{2})$.

Proof. Let us briefly recall – for readers convenience – how this lemma follows from Theorem 1 in [3]. By homogeneity we may assume that $r = 1$ and that ρ is fixed. Let $A' = (0, \ldots, 0, \frac{2\rho}{3})$ and let B_N denote the open ball $B_N(0,1)$ in $\mathbb{R}^N$. It is easy to construct a bi-Lipschitz map $F : U \to B_N(0,1)$ with a bi-Lipschitz constant depending only on ρ and N and which maps A' onto 0, $U \cap T(1/2)$ onto $B_N^- := \{x \in B_N;\ x_N < -\frac{1}{2}\}$ and $U \setminus T(\frac{3}{4})$ onto $B_N^+ := \{x \in B_N;\ x_N \geqslant \frac{1}{2}\}$.

Standard calculations show that if u is $\Delta - V$ harmonic in U then the function $u_1 := u \circ F^{-1}$ is $L_1 - V \circ F^{-1}$ harmonic in B_N for some (symmetric) divergence form elliptic operator $L_1 = \sum_{i,j} \partial_i(a_{ij}\partial_j)$ in B_N satisfying $C_1^{-1} I_N \leqslant \{a_{ij}\} \leqslant C_1 I_N$ with $C_1 = C_1(N, \frac{r}{\rho}) \geqslant 1$. Let $V_1 = V \circ F_1^{-1}$. Clearly $V_1 \in \mathcal{V}(B_N, a')$ for $a' = C^2 a$.

Other simple calculations show that the operator $\mathcal{L} = (1 - |x|)^2(L_1 - V_1)$ seen as a map $H^1_{loc}(B_N) \to H^{-1}_{loc}(B_N)$ is an adapted elliptic operator in divergence form over the hyperbolic ball B_N (i.e. w.r. to the hyperbolic metric $ds^2 = \frac{|dx|^2}{(1-|x|^2)^2}$) in the sense of [3]. Moreover since the form $\varphi \mapsto \int_{B_N} a_{ij}\partial_i\varphi\partial_j\varphi\,dx - \varepsilon_0 \int_{B_N} \frac{\varphi^2}{(1-|x|)^2}\,dx$ is coercive for $\varepsilon_0 = \varepsilon_0(C_1, N) > 0$ chosen sufficiently small, the differential operator $\mathcal{L}$ is weakly coercive which means that there exists $\varepsilon_0 = \varepsilon_0(N, \frac{r}{\rho}) > 0$ such that $\mathcal{L} + \varepsilon_0$ admits a Green's function in B_N.

This shows that Theorem 1 in [3] applies to $\mathcal{L}$. Thus there is a constant $c = c(\varepsilon_0, C_1, N)$, $c \geqslant 1$, such that for $z = (z', z_N) \in B_N^+$ and $y \in B_N^-$ one has

$$c^{-1} G_{\mathcal{L}}(y,z) \leqslant G_{\mathcal{L}}(y,0) G_{\mathcal{L}}(0,z) \leqslant c G_{\mathcal{L}}(y,z). \tag{A.5}$$

Here we have also used the standard Harnack inequalities for $\mathcal{L}$ and have denoted $G_{\mathcal{L}}$ the $\mathcal{L}$ Green's function in B_N w.r. to the hyperbolic metric (we adopt the notational convention that $u(x) := G_{\mathcal{L}}(x,y)$ satisfies $\mathcal{L}u = -\delta_x$ in the weak sense [26] w.r. to the hyperbolic volume). Notice that $G_{\mathcal{L}}(x,y) = \delta(y)^{N-2} g(x,y)$ if g is Green's function of $L_1 - V_1$ in B_N (w.r. to the usual metric).

Suppose that u_1 is positive $\mathcal{L}$ harmonic (i.e. $L_1 - V_1$ harmonic) in B_N and that u_1 vanishes on $\partial B_N \cap \{x \in \partial B_N;\ x_N \leqslant \frac{1}{2}\}$. Then u_1 can be represented as a Green potential in $B_N \cap \{x;\ x_N < \frac{1}{2}\}$: $u_1(y) = \int G_{\mathcal{L}}(y,z)\,d\nu(z)$ where ν is a nonnegative Borel measure on $\{z \in B_N;\ z_N = \frac{1}{2}\}$ and $y_N \leqslant \frac{1}{2}$. So upon integrating (A.5) we get (with another constant c)

$$c^{-1}u_1(y) \leqslant u_1(0)g(y,0) \leqslant cu_1(y) \tag{A.6}$$

for $y \in B_N^-$. Thus if u is a positive L_V solution in U that vanishes in $\partial_\# U$ it follows – on using the change of variable $y = F(x)$ – that

$$c^{-1}u(x) \leqslant u\big(A'\big)G\big(x, A'\big) \leqslant cu(x) \tag{A.7}$$

for $x \in U(\frac{1}{2})$, where G is Green's function w.r. to L_V in U. Using Harnack inequalities for L_V, the lemma easily follows. □

Remark. Using Lemma A.2, well-known arguments (see [2]) show that for every bounded Lipschitz domain Ω in $\mathbb{R}^N$ and every $V \in \mathcal{V}(\Omega, a)$, $a > 0$, the following potential theoretic properties hold in Ω equipped with $L_V := \Delta - V$ (we let G_y^V to denote the L_V Green's function in Ω with pole at y):

(a) For each $P \in \partial\Omega$, the limit $K_P^V(x) = \lim_{y \to P} G_y^V(x)/G_y^V(x_0)$, $x \in \Omega$, exists and K_P^V is a positive L_V-harmonic function K_P^V in Ω which depends continuously on P and vanishes continuously in $\partial\Omega \setminus \{P\}$.
(b) For each $P \in \partial\Omega$, every positive L_V-solution in Ω that vanishes on $\partial\Omega \setminus \{P\}$ is proportional to K_P^V.
(c) Every positive L_V-solution u in Ω can be written in a unique way as $u(x) = \int_{\partial\Omega} K_P^V(x)\,d\mu(P)$, $x \in \Omega$, for some positive (finite) measure μ in $\partial\Omega$. See [3].

A.3. Proof of Theorem A.1

Again Ω is a bounded Lipschitz domain in $\mathbb{R}^N$ and $V \in \mathcal{V}(\Omega, a)$, $a \geqslant 0$.

For the proof we use a simple variant of the comparison principle given in Lemma A.2. Notations are as before, in particular $U = U_f(r, \rho)$ is the domain considered in A2 and $A = A_U = (0, \ldots, 0, \frac{\rho}{2})$. Let $A' = (0, \ldots, 0, \frac{2\rho}{3})$.

Lemma A.3. *Let u be positive harmonic (w.r. to Δ) in U, let v be positive $\Delta - V$-harmonic in U and assume that $u = v = 0$ in $\partial_\# U$. Then*

$$\frac{v(x)}{v(A)} \leqslant c\frac{u(x)}{u(A)} \quad \text{for } x \in U \cap T\left(\frac{1}{2}\right) \tag{A.8}$$

for some positive constant c depending only on ρ/r, the constant a and N.

Proof. Assume as we may that ρ is fixed. We have seen that $v(x) \leqslant cv(A')G_{A'}^V(x)$ in $U \cap T(\frac{1}{2})$ and we know that $G_A^V \leqslant G_A^0$ in U if $G_{A'}^V$ is $(\Delta - V)$-Green's function in U with pole at A'. By maximum principle, Harnack inequalities and the known behavior of $G_{A'}^0$ in $B(A', \frac{r}{4})$

(more precisely $G^0_A(x) \leqslant c_1 := c_1(r, N)$ in $\partial B(A', \frac{r}{4})$) we have that $u(x) \geqslant c_1 v(A) G^0_{A'}(x)$ in $U \setminus B(A', \frac{r}{4})$. So that – using Harnack inequalities in $B(A', \frac{r}{2})$ for u and v – the lemma follows. □

Remark. The opposite estimate, i.e. $\frac{u(x)}{u(A)} \leqslant C \frac{v(x)}{v(A)}$ (with another constant $C > 0$), cannot be expected to hold in general as shown by simple (and obvious) examples.

Denote $g^V_{x_0}$ the Green's function with respect to $\Delta - V$ in Ω and with pole at x_0. For $y \in \partial\Omega$, a pseudo-normal for Ω at y is a unit vector $\nu \in \mathbb{R}^N$ such that for some small $\eta > 0$, the set $C(y, \nu_y, \eta) := \{y + t(\nu_y + v);\ 0 < t < \eta,\ \|v\| \leqslant \eta\}$ is contained in Ω.

Proposition A.4. *Given $y \in \partial\Omega$ and a pseudo-normal ν_y at y for U, the following assertions are equivalent*:

(i) $\tilde{K}^V_y = 0$ (*i.e.* $y \in \mathcal{S}ing_V(\Omega)$).
(ii) $\limsup_{t\downarrow 0} K^V_y(y + t\nu_y)/K_y(y + t\nu_y) = +\infty$.
(iii) $\lim_{t\downarrow 0} K^V_y(y + t\nu_y)/K_y(y + t\nu_y) = +\infty$.
(iv) $\lim_{x\to y} g^V_{x_0}(x)/g^0_{x_0}(x) = 0$.

Proof. (a) We first recall a standard consequence of Lemma A.2 that relates $g^V_{x_0}$ and K^V_y near y (for any $y \in \partial\Omega$).

Consider $u = K^V_y$ and $v := g^V_{y+t\nu_y}$. Using Lemma A.2 and the fact that $v \sim t^{2-N}$ in $\partial B(y + t\nu_y, \frac{\eta}{2}t)$, $0 < t < \eta$, we see that $u(x) \sim u(y + t\nu_y) t^{N-2} g^V_{y+t\nu_y}(x)$ for $x \in \Omega \setminus B(y + t\nu_y, t\eta/2)$ (here $\sim$ means "is in between two constant times" with constants depending only on y, Ω, ν_y and a).

Taking in particular $x = x_0$ we obtain that $K^V_y(y+t\nu_y) \sim 1/(t^{N-2} g^V(y+t\nu_y; x_0))$. In particular considering the special case $V = 0$, we get also that $K_y(y + t\nu_y) \sim 1/(t^{N-2} g(y + t\nu_y; x_0))$.

(b) Using the above we see that (ii) is equivalent to (iv)$'$: $\liminf_{t\downarrow 0} g^V_{x_0}(y + t\nu_y)/g^0_{x_0}(y + t\nu_y) = 0$.

(c) Now to show that (iv) and (iv)$'$ are equivalent we may assume that $y = 0$, $\nu_y = (0, \dots, 0, 1)$ and (with the notations above in A.2) that $T(1) \cap \Omega = U$, $U = U_f(r, \rho)$ and $x_0 \in \Omega \setminus \overline{U}$.

Applying Lemma A.3 to U, $u = g^V_{x_0}$, $v = g_{x_0}$, and $U_t = U_{t_j}$ for a sequence t_j, $t_j \downarrow 0$ such that $u(A_{t_j}) = o(v(A_{t_j}))$, $A_{t_j} = (0, \dots, 0, t_j)$, we get that $u(x) \leqslant c\frac{u(A_{t_j})}{v(A_{t_j})} v(x)$ in $\Omega \cap T(t_j \frac{\rho}{2})$. Hence (iv)$'$ implies (iv). And – using (a) again – conditions (ii), (iii) and (iv) are equivalent.

(d) Similarly if on the contrary $g^V(A_j, x_0) \geqslant c g(A_j, x_0)$, for some sequence $A_j = t_j \nu$, $t_j \downarrow 0$ and a positive real c, we have (since a priori $g^V \leqslant g$) that

$$K^V_{A_j}(x) := g^V(A_j, x)/g^V(A_j, x_0) \leqslant c^{-1} K_{A_j}(x) = c^{-1} g(A_j, x)/g(A_j, x_0) \tag{A.9}$$

and letting $j \to \infty$ we get $K^V_y \leqslant c^{-1} K_y$. Thus, (i) $\Rightarrow$ (iv).

Since obviously (ii) $\Rightarrow$ (i), Proposition A.4 is proved. □

The next lemma is the key for the proof of Theorem A.1. Returning again to the canonical Lipschitz domain $U = U_f(r, \rho)$, let $V \in \mathcal{V}_a(U)$ and for $\theta \in (0, \frac{1}{10})$, let $U^\theta := \{x \in U;\ d(x, \partial U) \geqslant \theta r\}$, $I^\theta_U := \int_{U^\theta} V(x) \frac{dx}{|\delta_U(x)|^{N-2}}$.

Obviously $\frac{1}{r^{N-2}}\int_{U^\theta} V(x)\,dx \leqslant I_U^\theta \leqslant \frac{1}{(\theta r)^{N-2}}\int_{U^\theta} V(x)\,dx$.

Lemma A.5. *Let u, $\tilde{u}$ be two nonnegative continuous functions in $\overline{U}$ that are respectively Δ-harmonic and L_V-harmonic in U. Assume that $\tilde{u} \leqslant u$ in ∂U and $\tilde{u} = u = 0$ in $\partial_\# U$. Then for some constant $c = c(\frac{r}{\rho}, a, \theta, N) > 0$,*

$$(1 + cI_\theta)\tilde{u}(x) \leqslant u(x) \quad \textit{for } x \in U \cap T\left(\frac{1}{2}\right). \tag{A.10}$$

Proof. Since the assumptions and the conclusion are invariant under dilations we may assume that r is fixed as well as ρ. Replacing u by the harmonic function in U with same boundary values as $\tilde{u}$ we may also assume that $u = \tilde{u}$ in ∂U. Since $\Delta(u - \tilde{u}) = -V\tilde{u}$ and $u - \tilde{u}$ vanishes on ∂U, we see that $u - \tilde{u} = G_U(V\tilde{u})$ where G_U is the usual Green's function in U.

By Harnack property and since $G_U(x, y) \geqslant c = c(\theta, a, N) > 0$ for $x \in B_1 = B(A_1, \frac{r}{100})$, $A_1 = (0, \dots, 0, \frac{3r}{4})$, and $y \in U^\theta$, we have

$$u(x) - \tilde{u}(x) \geqslant cI_\theta \tilde{u}(A_1), \quad x \in B_1.$$

Thus in U, $w(x) := u(x) - \tilde{u}(x) \geqslant cI_\theta \tilde{u}(A_1)R_1^{B_1}(x)$ where $R_1^{B_1}$ is the (classical) capacitary potential [10] of B_1 in U and using the comparison principle Lemma 1 for $V = 0$ we have $w \geqslant cI_\theta \tilde{u}(A_1)\frac{u}{u(A_1)}$ in $U(\frac{1}{2}) := T(\frac{1}{2}) \cap U$.

Using then Lemma A.2 (and Harnack inequalities)

$$w(x) \geqslant c'' I_\theta \tilde{u}(A_1)\frac{\tilde{u}(x)}{\tilde{u}(A_1)} = c''' I_\theta \tilde{u}(x), \quad x \in U\left(\frac{1}{2}\right).$$

Thus, $u(x) \geqslant (1 + c''' I_\theta)\tilde{u}(x)$ in $U(\frac{1}{2})$. $\square$

Proof of Theorem A.1. We may assume that $y = 0$, that for some r, ρ, f, $\Omega \cap T(1) = U := U_f(r, \rho)$ (with the notation fixed above in Section A2) and that $x_0 \notin \overline{U}$.

Set $T_n = T(2^{-n})$, $C_y^n := C_{\epsilon,y} \cap (T_n \setminus T_{n+1})$ for $n \geqslant 1$, $u = G_{x_0}^0$, $\tilde{u} = G_{x_0}^V$ (where $G_{x_0}^V$ is Green's function with pole at x_0 with respect to $\Delta - V$ in Ω). One may also observe that ε may be assumed so small that Σ_0^ε contains the truncated cone $C := \{(x', x_N);\ x_N < \frac{\rho}{2}, |x'| < \frac{r}{\rho}x_N\}$.

For each $n \geqslant 0$ there is a greatest $\alpha_n > 0$ such that $u \geqslant \alpha_n \tilde{u}$ in U_n (we know that $\alpha_n \leqslant 1$). By the key Lemma A.5 (and elementary geometric considerations)

$$\alpha_{n+1} \geqslant \alpha_n(1 + cI_{n+1}) \quad \text{if } I_m := \int_{C_m} \frac{V(x)}{\delta_\Omega(x)^{N-2}}\,dx \tag{A.11}$$

for some constant $c = c(\varepsilon, \frac{r}{\rho}, a, N)$ independent of n. Thus

$$\alpha_n \geqslant \alpha_0 \prod_{k=1}^{n-1}(1 + cI_k) \geqslant \alpha_0\left(1 + c\sum_{k=1}^{n-1} I_k\right) \geqslant c\alpha_0 \int_{C_1 \setminus C_{n+1}} \frac{V(x)}{\delta_\Omega(x)^{N-2}}\,dx$$

which shows that $\lim \alpha_n = +\infty$. Thus $G^V_{x_0} = o(G^0_{x_0})$ at y and by Proposition A.4 the point y belongs to $\mathcal{S}ing_V(\Omega)$. $\square$

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